FUNCTIONS

5 minute review. Recap the composition $g \circ f$ for functions $f : A \to B$ and $g : B \to C$, what it means for two functions $f, g : A \to A$ to commute and what it means for a function to have an inverse.

Class warm-up. Let $A = \mathbb{R}$ and $B = [0, \infty)$. Let $f : A \to B$ be given by $f(x) = x^2$ and $g : B \to A$ be given by $g(y) = \sqrt{y}$.

(a) Is $f \circ g = \mathrm{id}_B$? (b) Is $g \circ f = \mathrm{id}_A$? (c) Is g an inverse of f?

Problems. Choose from the below.

1. For fixed $a, b, c, d \in \mathbb{R}$, let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$ and $g(x) = cx + d$.
   (a) Let $a = 4$, $b = 2$, $c = 7$. Given that f and g commute, find d.
   (b) Now let $a = b = 1$ and let $c = 2$. Show that f and g do not commute for any value of d.
2. (a) Let $\alpha, \beta \in \mathbb{R}$. By showing that they have the same effect on a general point $(r, \theta) \in \mathbb{R}^2$, show that $\mathrm{rot}_\alpha \mathrm{ref}_\beta = \mathrm{ref}_{\alpha+\beta}$ and $\mathrm{ref}_\alpha \mathrm{rot}_\beta = \mathrm{ref}_{\alpha-\beta}$ as functions $\mathbb{R}^2 \to \mathbb{R}^2$. Can you prove these in more than one way?
   (b) In each of the following, find a rotation or reflection f which satisfies the given equation.
   (i) $f \circ \mathrm{ref}_{\frac{2\pi}{3}} = \mathrm{rot}_{\frac{\pi}{2}}$; (ii) $f \circ \mathrm{rot}_{\frac{2\pi}{3}} = \mathrm{ref}_{\frac{\pi}{6}}$; (iii) $f \circ \mathrm{ref}_{\frac{2\pi}{3}} = \mathrm{ref}_{\frac{\pi}{2}}$.
   (c) Show that ref_α commutes with the rotations rot_0 and rot_π, and find two reflections which commute with ref_α.
3. (a) Let $\alpha, \beta \in \mathbb{R}$. Show that $\mathrm{ref}_\alpha \mathrm{rot}_\beta = \mathrm{rot}_\beta \mathrm{ref}_\alpha$ if and only if $\mathrm{rot}_\beta = \mathrm{rot}_0$ or $\mathrm{rot}_\beta = \mathrm{rot}_\pi$. (Hint: you may have already proved one direction of this.)
   (b) Let $\alpha, \beta \in \mathbb{R}$. Show that $\mathrm{ref}_\alpha \mathrm{ref}_\beta = \mathrm{ref}_\beta \mathrm{ref}_\alpha$ if and only if $\mathrm{ref}_\beta = \mathrm{ref}_\alpha$ or $\mathrm{ref}_\beta = \mathrm{ref}_{\pi+\alpha}$.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$. The inverse trigonometric function $\sin^{-1}$ or arcsin has domain $[-1, 1]$ and codomain $[-\pi/2, \pi/2]$.
   (a) Give an example of a real number x such that $\sin^{-1}(\sin x) \neq x$. Is $\sin^{-1}$ an inverse for f?
   (b) Identify subsets A and B of $\mathbb{R}$ such that $\sin^{-1}$ is an inverse for the function $g : A \to B$ given by $g(a) = \sin(a)$.

For the warm-up, (a) yes: if $y \geq 0$, then $f \circ g(y) = (\sqrt{y})^2 = y$; (b) no: as a counterexample, $g \circ f(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$; (c) no, because of (b). Restricting the domain of f and the codomain of g to be B will make g into an inverse for f.

Selected answers and hints.

1. (a) Here $f \circ g(x) = f(7x+d) = 28x+4d+2$, whereas $g \circ f(x) = g(4x+2) = 28x+14+d$. Therefore f and g commute precisely when $4d+2 = 14+d$, that is when $d = 4$.

   (b) Here $f \circ g(x) = f(2x+d) = 2x+d+1$, whereas $g \circ f(x) = g(x+1) = 2x+2+d$. Therefore f and g commute precisely when $d+1 = 2+d$, that is for no value of d.

2. (a) Let $(r, \theta) \in \mathbb{R}^2$. Then $\mathrm{rot}_\alpha \mathrm{ref}_\beta(r,\theta) = \mathrm{rot}_\alpha(\mathrm{ref}_\beta(r,\theta)) = \mathrm{rot}_\alpha(r, \beta-\theta) = (r, \alpha+(\beta-\theta)) = (r, (\alpha+\beta)-\theta) = \mathrm{ref}_{\alpha+\beta}(r,\theta)$. Hence $\mathrm{rot}_\alpha \mathrm{ref}_\beta = \mathrm{ref}_{\alpha+\beta}$.

   Let $(r, \theta) \in \mathbb{R}^2$. Then $\mathrm{ref}_\alpha \mathrm{rot}_\beta(r,\theta) = \mathrm{ref}_\alpha(\mathrm{rot}_\beta(r,\theta)) = \mathrm{ref}_\alpha(r, \beta+\theta) = (r, \alpha-(\beta+\theta)) = (r, (\alpha-\beta)-\theta) = \mathrm{ref}_{\alpha-\beta}(r,\theta)$. Hence $\mathrm{ref}_\alpha \mathrm{rot}_\beta = \mathrm{ref}_{\alpha-\beta}$.

   Other ways: think of complex numbers $z = re^{i\theta}$ or try to draw a picture.

   (b) (i) $f = \mathrm{ref}_{\frac{7\pi}{6}}$; (ii) $f = \mathrm{ref}_{\frac{5\pi}{6}}$; (iii) $f = \mathrm{rot}_{\frac{-\pi}{6}} = \mathrm{rot}_{\frac{11\pi}{6}}$.

   (c) We have $\mathrm{ref}_\alpha \mathrm{rot}_0 = \mathrm{ref}_\alpha = \mathrm{rot}_0 \mathrm{ref}_\alpha$, so ref_α commutes with rot_0. Similarly, $\mathrm{ref}_\alpha \mathrm{rot}_\pi = \mathrm{ref}_{\alpha-\pi} = \mathrm{ref}_{\alpha-\pi+2\pi} = \mathrm{ref}_{\alpha+\pi} = \mathrm{rot}_\pi \mathrm{ref}_\alpha$, so ref_α commutes with rot_π.

   We are looking for two values of β for which $\mathrm{ref}_\alpha \mathrm{ref}_\beta = \mathrm{ref}_\beta \mathrm{ref}_\alpha$, that is $\mathrm{rot}_{\alpha-\beta} = \mathrm{rot}_{\beta-\alpha}$ (using the rot/ref formulae). The most obvious choice is when $\beta = \alpha$, but $\beta = \alpha+\pi$ also works, as $\mathrm{rot}_{\alpha-(\alpha+\pi)} = \mathrm{rot}_{-\pi} = \mathrm{rot}_\pi = \mathrm{rot}_{(\alpha+\pi)-\alpha}$. So two reflections that work are ref_α and $\mathrm{ref}_{\alpha+\pi}$. (Note that these are distinct reflections; that is, $\mathrm{ref}_\alpha \neq \mathrm{ref}_{\alpha+\pi}$.)

3. (a) In 3(c) above we've already shown that both rot_0 and rot_π commute with ref_α; that is, if $\mathrm{rot}_\beta = \mathrm{rot}_0$ or $\mathrm{rot}_\beta = \mathrm{rot}_\pi$ then $\mathrm{ref}_\alpha \mathrm{rot}_\beta = \mathrm{rot}_\beta \mathrm{ref}_\alpha$.

   For the converse, suppose that $\mathrm{ref}_\alpha \mathrm{rot}_\beta = \mathrm{rot}_\beta \mathrm{ref}_\alpha$. Then, by the rot/ref formulae, $\mathrm{ref}_{\alpha-\beta} = \mathrm{ref}_{\beta+\alpha}$. Hence $\beta+\alpha = \alpha-\beta+2n\pi$ for some integer n.[1] Therefore $\beta = n\pi$ for some integer n. If n is even then $\mathrm{rot}_\beta = \mathrm{rot}_0$ and if n is odd then $\mathrm{rot}_\beta = \mathrm{rot}_\pi$. Thus either $\mathrm{rot}_\beta = \mathrm{rot}_0$ or $\mathrm{rot}_\beta = \mathrm{rot}_\pi$.

   (b) In 3(c) we've shown that both ref_α and $\mathrm{ref}_{\alpha+\pi}$ commute with ref_α; that is, if $\mathrm{ref}_\beta = \mathrm{ref}_\alpha$ or $\mathrm{ref}_\beta = \mathrm{rot}_{\alpha+\pi}$ then $\mathrm{ref}_\alpha \mathrm{ref}_\beta = \mathrm{ref}_\beta \mathrm{ref}_\alpha$.

   For the converse, suppose that $\mathrm{ref}_\alpha \mathrm{ref}_\beta = \mathrm{ref}_\beta \mathrm{ref}_\alpha$. Then, by the rot/ref formulae, $\mathrm{rot}_{\alpha-\beta} = \mathrm{rot}_{\beta-\alpha}$. Hence $\beta-\alpha = \alpha-\beta+2n\pi$ for some integer n. Therefore $\beta = \alpha+n\pi$ for some integer n. If n is even then $\mathrm{ref}_\beta = \mathrm{ref}_\alpha$ and if n is odd then $\mathrm{ref}_\beta = \mathrm{ref}_{\alpha+\pi}$. Thus either $\mathrm{ref}_\beta = \mathrm{ref}_\alpha$ or $\mathrm{ref}_\beta = \mathrm{ref}_{\alpha+\pi}$.

4. (a) Take $x = \pi$. Then $\sin\pi = 0$ and $\sin^{-1}(0) = 0$ so $\sin^{-1}(\sin\pi) = 0 \neq \pi$. (Any value outside $[-\pi/2, \pi/2]$ will do.)

   (b) $A = [-\pi/2, \pi/2]$ and $B = [-1, 1]$.

[1] Important! Remember that we can't conclude here that $\beta+\alpha = \alpha-\beta$, but instead that they must differ by a multiple of 2π.