COUNTABILITY AND PERMUTATIONS

5 minute review. Recap the definition of a countable set, that is a set in bijective correspondence with $\mathbb{N} = \{1, 2, 3, \ldots\}$, and that this is equivalent to the set having a presentation $\{a_1, a_2, ...\}$ such that every element appears at some finite point on the list, and no element appears more than once. Also briefly cover the two-row notation for permutations, e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, and cycle notation, e.g. $(1\ 2\ 3)(4\ 5)$.

Class warm-up. Is $S = \{a \in \mathbb{R} : a^2 \in \mathbb{N}\}$ a countable set? What about the set $T = \{a \in \mathbb{R} : a^n \in \mathbb{N} \text{ for some } n \in \mathbb{N}\}$?

Problems. Choose from the below.

1. Show that each of the sets A_i below is countable, either by presenting it in the form $\{a_1, a_2, a_3, \ldots\}$, giving the terms as far as a_7, or by constructing an explicit bijection $f : \mathbb{N} \to A_i$.
   (a) $A_1 = \{a \in \mathbb{N} : a \text{ is odd}\}$.
   (b) $A_2 = \{a \in \mathbb{Z} : a \text{ is odd}\}$.
   (c) $A_3 = \{a \in \mathbb{R} : a = b^2 \text{ for some } b \in \mathbb{Z}\}$.
2. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 6 & 3 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix}$. Find $\alpha\beta, \beta\alpha$ and $\alpha^{-1}\beta$.
3. Recall that $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ consists of the remainders modulo 7.
   (a) Working in $\mathbb{Z}_7$, there is a permutation $\alpha_2 \in S_6$ for which $\overline{\alpha_2(i)} = \overline{2} \cdot \overline{i}$ for $1 \leq i \leq 6$. For example,
   $$\overline{\alpha_2(1)} = \overline{2} \cdot \overline{1} = \overline{2}, \text{ so } \alpha_2(1) = 2, \text{ and}$$
   $$\overline{\alpha_2(2)} = \overline{2} \cdot \overline{2} = \overline{4}, \text{ so } \alpha_2(2) = 4.$$
   Write down α_2 in two-row notation.
   (b) Again in $\mathbb{Z}_7$, there are permutations $\alpha_3, \alpha_4 \in S_6$ such that $\overline{\alpha_3(i)} = \overline{3} \cdot \overline{i}$ and $\overline{\alpha_4(i)} = \overline{4} \cdot \overline{i}$ for $1 \leq i \leq 6$. Express α_3 and α_4 in two row notation and verify that $\alpha_3^2 = \alpha_4^2 = \alpha_2$. Why do these identities hold?
   (c) Replace $\mathbb{Z}_7$ by $\mathbb{Z}_6$. Is there a permutation $\alpha \in S_5$ such $\overline{\alpha(i)} = \overline{2}\ \overline{i}$ for $1 \leq i \leq 5$? If not, what property of the number 7 makes things work?
4. Find a formula for the proportion, p_n, of permutations in S_n which are cycles (for $n \geq 2$). What happens to this proportion as n increases?

For the warm-up, $S = \{1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, 2, -2, \ldots\}$ is countable. The set T is also countable, which can be proved in an analogous way to that for $\mathbb{Q}$ by forming a 2×2 grid as below and snaking through, deleting repetitions.

$$\begin{array}{ccccccccc}
1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & \cdots \\
\sqrt{1} & -\sqrt{1} & \sqrt{2} & -\sqrt{2} & \sqrt{3} & -\sqrt{3} & \sqrt{4} & -\sqrt{4} & \cdots \\
\sqrt[3]{1} & \sqrt[3]{1} & \sqrt[3]{2} & \sqrt[3]{2} & \sqrt[3]{3} & \sqrt[3]{3} & \sqrt[3]{4} & \sqrt[3]{4} & \cdots \\
\sqrt[4]{1} & -\sqrt[4]{1} & \sqrt[4]{2} & -\sqrt[4]{2} & \sqrt[4]{3} & -\sqrt[4]{3} & \sqrt[4]{4} & -\sqrt[4]{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Selected answers and hints.

1. (a) $A_1 = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$ is countable. Alternatively, there's a bijection $f : \mathbb{N} \to A_1$ given by $f(i) = 2i - 1$ for all $i \in \mathbb{N}$.

   (b) $A_2 = \{1, -1, 3, -3, 5, -5, 7, \ldots\}$ is countable. Alternatively, there's a bijection $f : \mathbb{N} \to A_2$ given by $f(i) = \begin{cases} i & \text{if } i \text{ is odd} \\ 1 - i & \text{if } i \text{ is even.} \end{cases}$

   (c) $A_3 = \{0, 1, 4, 9, 16, 25, 36, 49, \ldots\}$ is countable. Alternatively, there's a bijection $f : \mathbb{N} \to A_1$ given by $f(i) = (i - 1)^2$ for all $i \in \mathbb{N}$.

2. $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 6 & 4 \end{pmatrix}$, $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 2 & 3 & 5 \end{pmatrix}$, $\alpha^{-1}\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}$.

3. (a) $\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix}$.

   (b) $\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}$ and $\alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 6 & 3 \end{pmatrix}$. To see why $\alpha_3^2 = \alpha_4^2 = \alpha_2$, notice that performing α_3 twice results in multiplication by $\overline{3} \cdot \overline{3} = \overline{9} = \overline{2}$. A similar thing occurs for α_4.

   (c) No. Here, we would get $\alpha(1) = 2, \alpha(2) = 4, \alpha(3) = 0, \alpha(4) = 2, \alpha(5) = 4$. This is not a bijective function on $\{1, 2, 3, 4, 5\}$. Even if we take $\{0, 1, 2, 3, 4, 5\}$ for the codomain, α would not be injective as $\alpha(1) = \alpha(4)$. The important difference is that 7 is prime whereas $6 = 2 \times 3$ is not.

4. There are $n!$ permutations in S_n in total. For $1 < k \leq n$, there are $\frac{n!}{(n-k)!k}$ cycles of length k, using the reasoning in Example 2.9 of the notes. There is only one 1-cycle (id). Thus, the number of permutations in S_n which are cycles is $\sum_{k=2}^{n} \frac{n!}{(n-k)!k} + 1$, so the proportion is $p_n = \sum_{k=2}^{n} \frac{1}{(n-k)!k} + \frac{1}{n!}$.

   The proportion of permutations which are cycles decreases as n increases (which seems about right intuitively, as more complex permutations become available with increasing n). Those who have learnt Python or another programming language might like to investigate numerically. To show this algebraically, one can compare the terms of p_n and p_{n+1}:

$$\begin{aligned}
p_n &= \left(\qquad \frac{1}{(n-1)!} \qquad + \frac{1}{2(n-2)!} + \ldots + \frac{1}{(n-1)1!} + \frac{1}{n} \right) + \frac{1}{n!} \\
&\geq \left(\left(\frac{1}{2(n-1)!} + \frac{1}{2(n-1)!} \right) + \frac{1}{3(n-2)!} + \ldots + \frac{1}{n.1!} + \frac{1}{n+1} \right) + \frac{1}{(n+1)!} \\
&\geq \left(\frac{1}{n(n-1)!} + \frac{1}{2(n-1)!} + \frac{1}{3(n-2)!} + \ldots + \frac{1}{n.1!} + \frac{1}{n+1} \right) + \frac{1}{(n+1)!} \\
&= p_{n+1}.
\end{aligned}$$

For more details, start a thread on the discussion board.