CYCLE DECOMPOSITIONS, PARITY AND ORDER

5 minute review. Demonstrate how to find the cycle decomposition of, for example, $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$ and $\beta = (1\ 2\ 3)(3\ 5)(2\ 3\ 5)(4\ 1\ 2)(1\ 2\ 3\ 4)$, and how to go from that to the permutation expressed as a product of transpositions. Also recap the definitions of odd and even permutations and the order of a permutation.

Class warm-up. Let $n \geq 2$ be a positive integer. Let $p_1 < \ldots < p_k$ be distinct primes ($k > 1$) with $p_1 + \ldots + p_k \leq n$. For (i)–(iii) below, is it possible to construct a permutation in S_n with the given order?

(i) p_1; (ii) $p_1 \times \ldots \times p_k$; (iii) p_1^k.

Problems. Choose from the below.

1. Find the cycle decompositions of each of the following permutations.

   (a) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 6 & 11 & 2 & 5 & 7 & 3 & 10 & 9 & 8 & 4 & 1 \end{pmatrix}$;

   (b) $\beta = (4\ 5\ 6)(3\ 4\ 5)(2\ 3\ 4)(1\ 2\ 3)$.

   Also express each of the permutations above as a product of transpositions, and hence write down its parity (even or odd) and its sign (± 1).

2. Let $\alpha = (1\ 9\ 3)(2\ 5\ 10)$, $\beta = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$ and $\gamma = (1\ 2)(3\ 4)(5\ 6\ 7\ 8)$. What are the orders of (i) α, (ii) β, (iii) γ, (iv) α^2, (v) β^{-1}, (vi) γ^0?

3. Express the transposition $(3\ 6)$ as a product of 5 adjacent transpositions $(a\ a+1)$ in two different ways.

4. **Bongard problems**[1]. In each case below, find a common property of the six objects on the left hand-side which none of the six objects on the right hand side has.

   (a)

abedc	pqrts	nmlok	jfghi
fhgij	vzxyw	xyvzw	tpqrs
xuvwt	mlkno	baecd	utwxv

   (b)

ABECD	BACED	ACBDE	DCEAB
ABCDE	EBADC	CBEAD	ABEDC
EABCD	CEADB	DABCE	EDBCA

5. A permutation is applied to the string SUPERBGOLDHAT. The same permutation is applied another four times to the output. The final output is BRHGLPESTAUDO. What was the first output?

[1]Google them!

For the warm-up,

(i) As $p_1 < n$, one can form the cycle $(1\ 2\ \ldots\ p_1)$.

(ii) One can compose disjoint cycles of lengths $p_1, \ldots, p_k$ to obtain such a permutation, e.g. $(1\ \cdots\ p_1)(p_1+1\ \cdots\ p_1+p_2)\cdots(\sum_{i=1}^{k-1} p_i + 1\ \cdots\ \sum_{i=1}^{k} p_i)$.

(iii) This depends on the value of n. For example, if $n = 5$ then $2 + 3 \leq n$ and $(1\ 2\ 3\ 4)$ has order 2^2, so it's possible here. On the other hand, if $n = 8$ then $3 + 5 \leq 8$ and there is no permutation in S_8 of order $3^2 = 9$, because each cycle in the cycle decomposition of such a permutation must have a length which is a factor of 9; since there are no 9-cycles in S_8, we must be left with cycles of length 3 only, in which case the permutation has order 3.

Selected answers and hints.

1. (a)
   (i) $\alpha = (1\ 12)(2\ 6\ 7\ 3\ 11\ 4)(5)(8\ 10)(9)$. Note that the 1-cycles (5) and (9) can be deleted.

   (ii) $\beta = (1\ 5\ 3)(2\ 6\ 4)$.

   (b) There are many possibilities here, but using the formulas in the notes,

   (i) $\alpha = (1\,12)(2\,6)(6\,7)(7\,3)(3\,11)(11\,4)(8\,10)$ or $(1\,12)(2\,4)(2\,11)(2\,3)(2\,7)(2\,6)(8\,10)$;

   (ii) $\beta = (1\ 5)(5\ 3)(2\ 6)(6\ 4)$ or $(1\ 3)(1\ 5)(2\ 4)(2\ 6)$.

2. The orders are (i) 3, (ii) 12, (iii) 4, (iv) 3, (v) 12, (vi) 1 (since any permutation raised to the power zero is the identity).

3. One solution comes from the formula in the notes: $(3\,6) = (5\,6)(4\,5)(3\,4)(4\,5)(5\,6)$. For a distinctly different formula, by thinking about the dynamics of what is happening in that first case (rip out some pieces of paper and shuffle them!), it's easy to spot an alternative, namely $(3\ 6) = (3\ 4)(4\ 5)(5\ 6)(4\ 5)(3\ 4)$.

4. (a) Those on the left all result from transpositions applied to a string of consecutive letters; none on the right have this property.

   (b) Those on the left all result from even permutations applied to a string of consecutive letters; none on the right have this property.

5. The permutation which sends SUPERBGOLDHAT to BRHGLPESTAUDO has cycle decomposition $(1\ 8\ 13\ 9\ 5\ 2\ 11\ 3\ 6)(4\ 7)(10\ 12)$. If α is the permutation which is being applied repeatedly, then this cycle decomposition represents α^5; that is, we are looking for $\alpha \in S_{13}$ such that $\alpha^5 = (1\ 8\ 13\ 9\ 5\ 2\ 11\ 3\ 6)(4\ 7)(10\ 12)$.

   It is not too hard to see that $\alpha = (1\ 13\ 5\ 11\ 6\ 8\ 9\ 2\ 3)(4\ 7)(10\ 12)$ works. Applying this permutation once will have given the output PLUGTHEBOARDS.

   To show that this solution is unique, consider the cycle decomposition for α. Any cycle of length k appearing here will lead to a cycle of length k in α^5 whenever $(5, k) = 1$. The remaining cases are $k = 5$ (which will result in the identity permutation) and $k = 10$ (which will give five 2-cycles). Hence the 9-cycle appearing in α^5 must have originated from a 9-cycle, and the two 2-cycles must have both started as 2-cycles. The result then follows.

   (There are neater solutions possible here: see if you can up with one, and post on the discussion board if you do!)

For more details, start a thread on the discussion board.