MAS439/MAS6320
CHAPTER 10: REGULAR RINGS

EVGENY SHINDER

A point $P \in X$ of an algebraic set X can be a singular or non-singular. For example for an algebraic curve $X = V(f)$, $f \in k[x, y]$ a point $P \in X$ is nonsingular if and only if the partial derivatives $\frac{\partial f}{\partial x}(P)$ and $\frac{\partial f}{\partial y}(P)$ are not both equal to zero.

To define the concept in general we first study the algebraic side - regular local rings. These are defined in terms of the tangent space to a ring: from the algebraic perspective a point is singular if the tangent space at this point has dimension bigger than the Krull dimension of the ring. We then relate the algebraic definition to non-vanishing of the Jacobian matrix of derivatives.

10.1. **Tangent spaces and regular local rings.** Let R be a ring, and and let $m \subset R$ be a maximal ideal with quotient field $k = R/m$. Consider an R-module $V = m/m^2$ with an obvious action $r \cdot \overline{x} = \overline{rx}$. Since m annihilates V, V is an R/m-module, i.e. a vector space over k.

Definition 10.1. *The k-vector space m/m^2 is called the cotangent space to R at m, and the dual k-vector space $T_{m,R} = (m/m^2)^* = \mathrm{Hom}(m/m^2, k)$ is called the tangent space to R at m.*

Theorem 10.2. *Let (R, m) be a local Noetherian ring. Then*
1. $\dim T_{m,R} =$ *minimal number of generators of* $m \subset R$
2. $\dim T_{m,R} \geq \dim R$

Proof. 1. Let $k = R/m$, $n = \dim T_{m,R} = \dim_k m/m^2$. Then clearly m can not be generated by less than n elements. We will now show that m can be generated by n elements.

We apply Theorem 4.9 (c) (Nakayama's Lemma) to R-module $M = m$ and ideal $I = m$: pick a k-basis $\overline{f_1}, \dots, \overline{f_n}$ of m/m^2. These elements are also generators of m/m^2 as R-module. By Theorem 4.9 (c) their preimages $f_1, \dots, f_n \in m$ generate m as R-module. (Here we used the assumption that R is Noetherian: Nakayama's Lemma is a statement about Noetherian rings.)

2. Let $n = \dim T_{m,R}$. Part (1) implies that m is generated by n elements $f_1, \dots, f_n \in m$. Now by Krull's principal ideal theorem (Theorem 10.3 below) applied inductively we get

$$\dim R/(f_1, \dots, f_j) \geq \dim R - j.$$

In particular, for $j = n$ we get

$$0 = \dim R/m \geq \dim R - n$$

so that $n = \dim T_{m,R} \geq \dim R$. □

Theorem 10.3 (Krull's principal ideal theorem, Algebraic form). *Let R be an a Noetherian ring, and let $f \in R$. Then exactly one of the following conditions holds:*
1. $R/(f) = 0$ *and f is a unit in R*
2. $\dim(R/(f)) = \dim(R)$, *and f is a zero-divisor in R*
3. $\dim(R/(f)) = \dim(R) - 1$

Date: May 5, 2016.

Corollary 10.4 (Krull's principal ideal theorem, Geometric form). *Let X be an algebraic set, $f \in k[X]$, and $Y = V(f) \subset X$, the zero locus of the function f. Then $\dim(Y)$ is equal to $\dim(X) - 1$ or $\dim(X)$, unless Y is an empty set.*

Example 10.5. *$X = V(xy) \subset \mathbb{A}^2$ has dimension one. The three possibilities of the Theorem are realized as follows:*

(1) *$f = 1$, $V(f) = \emptyset$*
(2) *$f = x$, $V(f) = V(x) \subset V(xy)$ have same dimension one*
(3) *$f = x^2 - 4$, $V(f)$ consists of two points $(2, 0)$ and $(-2, 0)$ and has dimension zero which is one less than dimension of X*

Definition 10.6. *A local ring (R, m) is called* **regular** *if $\dim R = \dim T_{m,R}$.*

Example 10.7. *We consider three local rings: $\mathbb{Z}_{(7)}$, k (a field) and $k[x]/(x^2)$ and investigate which ones among these are regular.*

ring R	m	$\dim(R)$	$\dim(T_{m,R})$	*regular?*
$\mathbb{Z}_{(7)}$	$(\frac{7}{1})$	1	1	*yes*
k	(0)	0	0	*yes*
$k[x]/(x^2)$	(x)	0	1	*no*

10.2. **Tangent spaces and regular rings in the non-local case.**

Definition 10.8. *A ring R is called* **regular** *if for every maximal $m \subset R$ the localization R_m is regular.*

For example $R = \mathbb{Z}$ is a regular ring, since all its localizations at maximal ideals $\mathbb{Z}_{(p)}$ are regular local rings.

Lemma 10.9. *Let (R_m, m_m) be localization of a ring R at a maximal ideal $m \subset R$. Then there are natural isomorphisms $k = R_m/m_m \simeq R/m$ as fields, and $m_m/m_m^2 \simeq m/m^2$ as k-vector spaces.*

Proof. We recall that in the Chapter on localization we have proved that for $U \subset R$ a multiplicative set and $N \subset M$ modules we have $U^{-1}(M/N) = U^{-1}M/U^{-1}N$ as $U^{-1}R$-modules. We apply this to $U = R\backslash m$ to obtain

$$R_m/m_m = (R/m)_m = k$$
$$m_m/m_m^2 = (m/m^2)_m = m/m^2.$$

The localization in the second step does not change the modules as these are $k = R/m$-modules, so the complement to m already acts by invertible elements. ☐

Proposition 10.10. *The polynomial ring $R = k[x_1, \dots, x_n]$ is a regular ring.*

Proof. To simplify the argument we give a proof in the case when k is algebraically closed.

We know that $\dim(R) = n$. Using Theorem 10.2 it suffices to check that every maximal ideal is generated by n elements.

By a corollary from the Hilbert Nullstellensatz we know that maximal ideals in R have the form

$$m = m_a = (x_1 - a_1, \dots, x_n - a_n),\ a \in k^n$$

and the result follows. ☐

Remark 10.11. *Let us investigate the tangent space to $k[x_1, \dots, x_n]$ at $m = m_a$, $a \in k^n$ in some detail.*

Let $y_i = x_i - a_i$. We have

$$f \in m \iff f(y_1,\dots,y_n) = \sum_{i=1}^{n} A_i y_i + \sum_{i,j=1}^{n} B_{ij} y_i y_j + \text{higher order terms}$$

$$f \in m^2 \iff f(y_1,\dots,y_n) = \sum_{i,j=1}^{n} B_{ij} y_i y_j + \text{higher order terms.}$$

Hence m/m^2 consists of $A_1\overline{y_1}+\dots+A_n\overline{y_n}$, $A_i \in k$, and has dimension n. We will write $dx_i = \overline{y_i}$, so that the dual space $(m/m^2)^*$ is spanned by "directional derivatives" $\frac{\partial}{\partial x_i}$.

The meaning of this notation can be explained as follows: elements of the tangent space k-linear maps from m/m^2 to k, and $\frac{\partial}{\partial x_i}$ correspond to maps:

$$f \in m/m^2 \mapsto \frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i}(a).$$

10.3. **Nonsingular algebraic sets.**

Definition 10.12. *An algebraic set X is called* nonsingular *at a point P if the local ring $k[X]_{m_P}$ is regular.*

Definition 10.13. *An algebraic set X is called* nonsingular *if it is nonsingular at every point $P \in X$, or equivalently, if the ring $k[X]$ is regular.*

To make sense of these definitions we start with explaining the geometric meaning of the tangent space.

Proposition 10.14. *Let $X \subset \mathbb{A}^n$ be an algebraic set with ideal $I(X) = \langle f_1,\dots,f_r\rangle$. The tangent space to X at $a \in X$ can be identified with the following vector subspace of k^n:*

$$\{(\alpha_1,\dots,\alpha_n) \in k^n \mid \frac{\partial f_i(a)}{\partial x_1}\alpha_1 + \dots + \frac{\partial f_i(a)}{\partial x_n}\alpha_n = 0 \ \forall i = 1\dots r\}.$$

Proof. We consider m/m^2, and then take the dual vector space.

We change the coordinates to $y_j = x_j - a_j$ and write Taylor expansions for every f_i:

$$f_i(y_1,\dots,y_n) = 0 + \frac{\partial f_i(a)}{\partial x_1} y_1 + \dots + \frac{\partial f_i(a)}{\partial x_n} y_n + O(y^2).$$

Computing $\overline{m_a}/\overline{m_a}^2 = (y_1,\dots,y_n)/((y_i y_j)_{i,j=1}^n + I(X))$ yields a quotient k-vector space

$$\frac{k dy_1 \oplus \dots \oplus k dy_n}{(\frac{\partial f_i(a)}{\partial x_1} dy_1 + \dots + \frac{\partial f_i(a)}{\partial x_n} dy_n)_{i=1}^r} = V/L$$

and the dual space is the one we need:

$$(V/L)^* = L^\perp = \{(\alpha_1,\dots,\alpha_n) : \sum_{i=1}^{n} \frac{\partial f_j(a)}{\partial x_i}\alpha_i = 0 \text{ for all } j\} \subset V^* = k^n.$$

□

Example 10.15. *Let $X = V(f) \subset \mathbb{A}^n$, i.e. X is given by one equation*

$$f(x_1,\dots,x_n) = 0.$$

Assume that X is non-empty. The tangent space to X at $a \in X$ has the form

$$T_{a,X} = \{(\alpha_1, \dots, \alpha_n) \in k^n : \frac{\partial f(a)}{\partial x_1}\alpha_1 + \dots + \frac{\partial f(a)}{\partial x_n}\alpha_n = 0\} \subset k^n.$$

Thus there are two cases: $T_{a,X} = k^n$ (when all partial derivatives of f are zero at a), or $T_{a,X}$ has dimension $n-1$ (when at least one of the partial derivatives of f is not zero at a).

By Krull's Principal Ideal Theorem (Theorem 10.3) we have $\dim X = \dim(R_X)_m = n-1$ for every maximal ideal $m \subset R_X$. It follows from the description of the tangent space $T_{a,X}$ given above that X is nonsingular if and only if for every $a \in X$ the derivatives $\partial f/\partial x_i(a)$ do not vanish simultaneously.

Example 10.16. *Computing the partial derivatives we see that $y = x^2$ is a nonsingular curve, and $y^2 = x^3$ is singular at $(0,0)$.*

Proposition 10.17 (Jacobian criterion)**.** *Let $X = V(f_1, \dots, f_r) \subset \mathbb{A}^n$ be an algebraic set. If the Jacobian matrix*

$$J = (\frac{\partial f_i}{\partial x_j}(a))_{i=1\dots r, j=1\dots n}$$

has rank r for every $a \in X$, then X is non-singular of dimension $n-r$.

Proof. Note that if $X \neq \emptyset$, then by Krull's principal ideal theorem we have $\dim(X) \geq n-r$. We've seen that the tangent space $T_{a,X}$ to X at a is given by the kernel of $J : k^n \to k^r$. Since J has rank r, the kernel has dimension $n-r$. Thus we have

$$\dim T_{a,X} = n - r \leq \dim X$$

and the converse inequality holds always. □