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CHAPTER 3: LOCALIZATION

EVGENY SHINDER (PARTLY BASED ON NOTES BY JAMES CRANCH)

The concept of localization generalizes the fraction field construction, i.e. the process of forming $\mathbb{Q}$ from $\mathbb{Z}$. This applies equally well to rings and modules. Geometrically localization corresponds to considering small open neighbourhoods in algebraic sets, hence the name.

Denominators for localization must lie in a so-called multiplicative subset:

Definition 3.1. *A* multiplicative subset U *of a ring* R *is a subset* $U \subset R$ *such that:*

- $1 \in U$, *and*
- *if* $a, b \in U$, *then* $ab \in U$.

Example 3.2. *If* $f \in R$ *is an element, then*

$$\{1, f, f^2, f^3, \dots\}$$

is a multiplicative subset.

Example 3.3. *Let* $U = R\backslash\{0\}$. *Then* U *is a multiplicative subset if and only if* R *is an integral domain.*

Example 3.4. *Let* $I \subset R$ *be an ideal. Then* $U = R\backslash I$ *is a multiplicative subset if and only* I *is a prime ideal. This example generalizes the one before:* R *is an integral domain if and only if* $\{0\}$ *is a prime ideal.*

Now that we understand multiplicative subsets, we can use them to define rings of fractions. Given a multiplicative subset U of R, we define a new ring $U^{-1}R$ to have as elements symbols of the form $\frac{a}{b}$, where $a \in R$ and $b \in U$, subject to an equivalence relation.

The correct equivalence relation which we'll use might be thought a little surprising: we say that $\frac{a}{b} = \frac{c}{d}$ if there is $t \in U$ such that

$$t(ad - bc) = 0$$

or equivalently

$$tad = tbc.$$

We ought to make the following check:

Proposition 3.5. *The relation on fractions defined above is an equivalence relation.*

Proof. Reflexivity is easy: $\frac{a}{b} = \frac{a}{b}$ as we can take $t = 1$ and get $1ab = 1ab$. Symmetry is similarly clear: if $tad = tbc$ then $tbc = tad$.

To check transitivity suppose $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$. This means that there is some $t \in U$ such that $tad = tbc$, and some $u \in U$ such that $ucf = ude$. We need to find $v \in U$ such that $vaf = vbe$. The obvious choice is $v = tud$ (which is in U, as all the factors are), and then

$$vaf = tuadf = tubcf = tubde = vbe$$

as required. □

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Now we show that this structure is a ring:

Theorem 3.6. *The set $U^{-1}R$, under the equivalence relation above, equipped with the usual arithmetic operations for fractions, is a commutative ring.*

Proof. What we really need to show that the usual operations are *well-defined*: that is, equivalent inputs give equivalent outputs.

First we show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$, or in other words that $\frac{af+be}{bf} = \frac{cf+de}{df}$.

So we have $tad = tbc$ for some $t \in U$, and we need to show that for some $u \in U$ we have $u(af + be)df = u(cf + de)bf$. If we take $u = t$ we get

$$t(af + be)df = tadf^2 + tbdef = tbcf^2 + tbdef = t(cf + de)bf$$

as needed.

Secondly, we show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b}\frac{e}{f} = \frac{c}{d}\frac{e}{f}$, or in other words that $\frac{ae}{bf} = \frac{ce}{df}$. So, again we have $tad = tbc$ for some $t \in U$, and we need to show that for some $u \in U$ we have $uadef = ubcef$. Again this works with $u = t$.

This having been proved, the rest (the commutative ring axioms) depends simply on standard properties of fraction arithmetic. □

There is a ring homomorphism $L_U : R \to U^{-1}R$ sending $a \mapsto \frac{a}{1}$, generalising the map $Z \to \mathbb{Q}$. Note that under this homomorphism, everything in U becomes invertible in $U^{-1}R$, since $L_U(u) = \frac{u}{1}$ has inverse $\frac{1}{u}$.

This observation gives us a way of explaining this construction, which is less explicit but often more helpful:

Theorem 3.7. *Let $f : R \to S$ be a ring homomorphism which sends elements of U to invertible elements in S. There is a unique ring homomorphism $g : U^{-1}R \to S$ such that f is the composite gL_U.*

Proof. We must have $g(\frac{a}{1}) = f(a)$ for $a \in R$, and we must have $g(\frac{1}{b}) = f(b)^{-1}$ for $b \in U$. Therefore we must have

$$g\left(\frac{a}{b}\right) = g\left(\frac{a}{1}\frac{1}{b}\right) = g\left(\frac{a}{1}\right) g\left(\frac{1}{b}\right) = f(a)f(b)^{-1}.$$

As a result of this, we know what g must be: so there is at most one homomorphism; we merely need to show that this works.

It sends equal elements to equal elements: if $\frac{a}{b} = \frac{c}{d}$ then there is some $t \in U$ with $tad = tbc$. Under these circumstances, g sends $\frac{a}{b}$ to $f(a)f(b)^{-1}$ and $\frac{c}{d}$ to $f(c)f(d)^{-1}$.

These are, however, equal:

$$\begin{aligned} f(a)f(b)^{-1} &= f(a)f(d)f(d)^{-1}f(t)f(t)^{-1}f(b)^{-1} \\ &= f(tad)\left(f(d)f(t)f(b)\right)^{-1} \\ &= f(tbc)\left(f(d)f(t)f(b)\right)^{-1} \\ &= f(c)f(b)f(b)^{-1}f(t)f(t)^{-1}f(d)^{-1} \\ &= f(c)f(d)^{-1}. \end{aligned}$$

Now we know that it is well-defined, it is easy to check that g is a homomorphism. □

We like to say that $U^{-1}R$ is the *universal* commutative ring with a homomorphism from R which sends elements of U to invertible elements.

As opposed to the familiar case $\mathbb{Z} \to \mathbb{Q}$ the localization homomorphism L_U is not always injective:

Lemma 3.8. *The kernel of L_U has the form:*

$$\mathrm{Ker}(L_U) = \{r \in R : t \cdot r = 0, \textit{ for some } t \in U\}.$$

Proof. The proof is straightforward:

$$r \in \mathrm{Ker}(L_U) \iff \frac{r}{1} = \frac{0}{1} \in U^{-1}R \iff t \cdot r = 0, \text{ for some } t \in U.$$

□

Corollary 3.9. *If R is an integral domain, and $U \subset R$ a multiplicative system in R, then the homomorphism $L_U : R \to U^{-1}R$ is injective. So we'll simply write $R \subset U^{-1}R$ in this case.*

With all these preliminaries in place, we can define the central thing of interest:

Definition 3.10. *Let R be a commutative ring, and let P be a prime ideal in R. The* localization *of R at P, written R_P, is the commutative ring $(R \backslash P)^{-1}R$.*

Unpacking this definition, we can consider R_P to be a ring of fractions $\frac{a}{b}$, where $a, b \in R$ and $b \notin P$.

Example 3.11 (Localization in Number Theory)**.** *Take $R = \mathbb{Z}$ and $P = (p)$ where p is a prime. For the sake of explicitness, let's take $p = 7$.*

The localization $\mathbb{Z}_{(7)}$ is the set of fractions $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $7 \nmid b$. It's a subring of $\mathbb{Q}$: we can add, subtract, and multiply, and we can divide by any prime except 7. The effect of this is to make a ring where 7 is the only prime: all other primes have become units.

Number theorists use these rings (for values of p which may or may not be 7) a lot to focus on one prime at a time. They call them the p-local integers.

Example 3.12 (Geometric meaning of localization)**.** *In the world of Algebraic Geometry the ring $\mathbb{C}[z]$ represents the ring of functions on the complex line $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$.*

The field of rational functions

$$\mathbb{C}(z) = (\mathbb{C}[z] \backslash \{0\})^{-1}\mathbb{C}[z] = \Big\{ \frac{f(z)}{g(z)} : f, g \in \mathbb{C}[z] \Big\}$$

represents meromorphic functions, i.e. functions which admit poles: one for every root of $g(z)$.

Now let us take a prime ideal $P = (z - a)$ and see what localization at P does to $\mathbb{C}[z]$:

$$\mathbb{C}[z]_P = \Big\{ \frac{f(z)}{g(z)} : f \in \mathbb{C}[z], g \in \mathbb{C}[z] \backslash (z - a) \Big\} = \Big\{ \frac{f(z)}{g(z)} : f, g \in \mathbb{C}[z], g(a) \neq 0 \Big\}$$

This is precisely the ring of meromorphic functions with no poles at $z = a$, i.e. functions regular at a. In other words when considering localization ring $\mathbb{C}[z]_P$ we restrict our attention to a small open neighbourhood around $z = a$.

The following theorem explains that localization of rings is a useful tool for similar reasons to why quotient rings are useful: these constructions make rings simpler, in particular there are fewer prime ideals in the localized ring than in the original ring (think of the $\mathbb{Z}_{(7)}$ example above!).

Theorem 3.13. *Let $U \subset R$ be a multiplicative subset. Then we have an inclusion-preserving bijection:*

$$\{\text{Prime ideals } P' \subset U^{-1}R\} \leftrightarrow \{\text{Prime ideals } P \subset R \text{ such that } P \cap U = \emptyset\} \tag{3.1}$$

Under this bijection maximal ideals correspond to maximal ideals.

Proof. The map from the set of ideals on the left to the set of ideals on the right is $P' \mapsto L_U^{-1}(P')$. Note that the preimage of a prime ideal under any ring homomorphism is a prime ideal. The ideal $L_U^{-1}(P')$ does not intersect U, otherwise P' would contain an image of an element of U, which is impossible as elements of U become units in $U^{-1}R$.

It is also clear that this map preserves inclusions: $P_1' \subset P_2' \implies L_U^{-1}P_1 \subset L_U^{-1}P_2$.

The map in other direction attaches to an ideal P the ideal

$$U^{-1}P = \left\{ \frac{a}{s} : a \in P, s \in U \right\}.$$

One easily checks that this is indeed an ideal. To see that $U^{-1}P$ is prime consider a product

$$\frac{a}{s} \cdot \frac{a'}{s'} \in U^{-1}P,$$

so that

$$\frac{aa'}{ss'} = \frac{a}{s} \cdot \frac{a'}{s'} = \frac{b}{u}$$

with $b \in P$. This yields

$$t(uaa' - ss'b) = 0 \implies (tu)aa' = tss'b \in P,$$

and since P is prime one of the elements tu, a or a' is in P. But since $P \cap U = \emptyset$, we deduce that $tu \notin P$, so that either a or a' belongs to P, which means that one of the original fractions

$$\frac{a}{s}, \frac{a'}{s'}$$

belongs to P.

It is clear that the assignment $P \mapsto U^{-1}P$ preserves inclusions:

$$P_1 \subset P_2 \implies U^{-1}P_1 \subset U^{-1}P_2.$$

It remains to check that the two maps just defined are inverses of each other: this will establish the desired bijection.

We first take a prime ideal $P \subset R$, $P \cap R = \emptyset$ and consider

$$\begin{aligned} L_U^{-1}(U^{-1}P) &= L_U^{-1}\left(\left\{\frac{a}{s} : a \in P\right\}\right) = \\ &= \left\{ b \in R : L_U(b) = \frac{a}{s}, \text{ for some } a \in P,\, s \in U \right\} = \\ &= \left\{ b \in R : \frac{b}{1} = \frac{a}{s}, \text{ for some } a \in P,\, s \in U \right\} = \\ &= \{ b \in R : t(bs - a) = 0, \text{ for some } a \in P,\, t, s \in U \} = \\ &= \{ b \in R : ub \in P, \text{ for some } u \in U \} = P. \end{aligned}$$

Now we take a prime ideal $P' \subset U^{-1}R$ and consider

$$U^{-1}L_U^{-1}(P') = U^{-1}\left\{a \in R : \frac{a}{1} \in P'\right\} =$$
$$= \left\{\frac{a}{u} \in R : \frac{a}{1} \in P', u \in U\right\} = P'$$

□

Corollary 3.14. *If $P \subset R$ is a prime ideal, then the ring R_P has the unique maximal ideal*

$$m_P = \{\frac{p}{t} : p \in P, t \notin P\} \subset R_P,$$

and prime ideals of R_P are in bijection with prime ideals of R contained in P.

Proof. The previous Theorem tells us that prime ideals in R_P are in inclusion-preserving bijection with ideals in R which do not intersect with $R\backslash P$, i.e. are contained in P.

Among these the maximal ideal is P, and it corresponds to $m_P = U^{-1}P$. □

Definition 3.15. *A ring R with a unique maximal ideal is called a local ring.*

Thus the previous Corollary tells us that localized rings R_P are local. Examples are rings $\mathbb{Z}_{(7)}$ and $\mathbb{C}[x]_{(x-a)}$ considered earlier.

3.1. **Localization of modules.** Given a ring R, a multiplicative subset U and a R-module M, we can use fractions to form a $U^{-1}R$-module $U^{-1}M$: objects are fractions $\frac{a}{b}$ with $a \in M$ and $b \in U$, subject to the equivalence relation that $\frac{a}{b} = \frac{c}{d}$ if there is $t \in U$ such that $tad = tbc$.

Addition is defined by the usual formula:

$$\frac{a}{b} + \frac{c}{d} = \frac{da + bc}{bd},$$

(note that this does make sense: we're never multiplying two elements of M together), and scalar multiplication by elements of $U^{-1}R$ is defined by the usual formula

$$\frac{r}{s} \cdot \frac{a}{b} = \frac{ra}{sb}.$$

It is not hard to check that these definitions work.

It's also easy to show that we have

$$U^{-1}(M \oplus N) = U^{-1}M \oplus U^{-1}N. \tag{3.2}$$

Example 3.16. *Let U be the multiplicative subset $\mathbb{Z}\backslash\{0\} \subset \mathbb{Z}$. Then we have $\mathbb{Q} = U^{-1}\mathbb{Z}$.*

Now suppose we have an abelian group A, and regard it as a $\mathbb{Z}$-module. Hence $U^{-1}A$ is supposed to be a $\mathbb{Q}$-module: a rational vector space. We can explain how this works in the case where A is finitely generated. In this case, $A = \mathbb{Z}^n \oplus A'$, where A' is finite.

For any finite abelian group A', we have $U^{-1}A' = 0$, the trivial vector space! Indeed, any element $x \in A'$ has $kx = 0$ for some nonzero $k \in \mathbb{Z}$, and then

$$\frac{x}{1} = \frac{0}{1} \qquad \text{since } kx1 = k01.$$

On the other hand, considering the free $\mathbb{Z}$-module $\mathbb{Z}$, we have $U^{-1}\mathbb{Z} = \mathbb{Q}$. Indeed, the definition is the same as the standard definition of $\mathbb{Q}$ as the fraction field of $\mathbb{Z}$.

Hence, for this particular choice of U, we have

$$U^{-1}A = U^{-1}(\mathbb{Z}^n \oplus A') = \mathbb{Q}^n;$$

the rule is "throw away the torsion and turn the $\mathbb{Z}$s into $\mathbb{Q}$'s".

One extra feature, which is frequently important, is this:

Proposition 3.17. *Let R be a commutative ring, U a multiplicative subset, and M and N both R-modules, and let $\phi : M \to N$ be an R-module homomorphism. Then there is a $U^{-1}R$-module homomorphism $U^{-1}\phi : U^{-1}M \to U^{-1}N$.*

Proof. Define

$$U^{-1}\phi\left(\frac{a}{b}\right) = \frac{\phi(a)}{b};$$

it is immediate to check that this works. □

We call this behaviour *functoriality*: the phenomenon that many constructions that can be done to objects can also be done to their homomorphisms. Formally this means that localization is a *functor*

$$U^{-1} : R - mod \to (U^{-1}R) - mod \tag{3.3}$$

between the category of R-modules and the category of $U^{-1}R$-modules.

Definition 3.18. *Let R be a ring, P a prime ideal of R and M an R-module. We define the* localization M_P *of M to be the R_P-module $(R\backslash P)^{-1}M$.*

It is not hard to imagine that this should be an important tool: if localization lets us look at a commutative ring R "one prime ideal at a time", then localising modules allows us to look at the effects of each prime ideal of R on their modules, too.

By the above, localization of modules is functorial: if we have R a commutative ring, P a prime ideal, and $M \to N$ a homomorphism of R-modules, then we get a homomorphism of P-modules $M_P \to N_P$.

We now investigate properties of the localization functor (3.3). From (3.2) we know that it preserves direct sums. We check that it also preserves kernels and quotients.

Proposition 3.19. *Let $U \subset R$ be a multiplicative subset and $N \subset M$ a submodule of R-module. Then we have a natural isomorphism*

$$U^{-1}(M/N) \simeq (U^{-1}M)/(U^{-1}N).$$

Digression on cokernels. Did it ever occur to you that definitions of injective and surjective as usually given are not exactly analogous? Here is a way to make the definitions of injective and surjective match up.

Let $f : M \to N$ be a homomorphism of R-modules. Then we define cokernel of f as the quotient:

$$\mathrm{Coker}(f) = N/\mathrm{I}m(f).$$

Now we have:

$$f \text{ injective} \iff \mathrm{Ker}(f) = 0$$
$$f \text{ surjective} \iff \mathrm{Coker}(f) = 0$$

Proposition 3.20. *Localization functor (3.3) preserves kernels, images and cokernels, that is if $f : M \to N$ is a homomorphism of R-modules, then there are natural isomorphisms*

$$U^{-1}\,\mathrm{Ker}(f) \simeq \mathrm{Ker}(U^{-1}f)$$
$$U^{-1}\mathrm{I}m(f) \simeq \mathrm{I}m(U^{-1}f)$$
$$U^{-1}\,\mathrm{Coker}(f) \simeq \mathrm{Coker}(U^{-1}f)$$

where $U^{-1}f : U^{-1}M \to U^{-1}N$.

Proof. For the kernel we calculate:

$$\begin{aligned}\mathrm{Ker}(U^{-1}f) &= \left\{\frac{m}{u} : U^{-1}f\left(\frac{m}{u}\right) = \frac{0}{1}\right\} = \\ &= \left\{\frac{m}{u} : \frac{f(m)}{u} = \frac{0}{1}\right\} = \\ &= \left\{\frac{m}{u} : tf(m) = 0, \text{ for some } t \in U\right\} = \\ &= \left\{\frac{m}{u} : f(m) = 0\right\} = U^{-1}Ker(f)\end{aligned}$$

and now for the image:

$$\mathrm{I}m(U^{-1}f) = \left\{\frac{f(m)}{u} : m \in M, u \in U\right\} = U^{-1}\mathrm{I}m(f).$$

Finally for the cokernel the statement follows from the one about the image and Proposition 3.19. ☐