MAS439/MAS6320
CHAPTER 6: NOETHER NORMALIZATION

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6.1. **Noether Normalization.**

Theorem 6.1 (Noether Normalization, geometric form)**.** *Let k be an algebraically closed field, and let X be an algebraic set. Then there exists an $n \geq 0$ and a finite surjective polynomial map $\phi : X \to \mathbb{A}^n$.*

Remark 6.2. *Recall that a finite polynomial map has finite fibers, so it's reasonable to expect that in the setup of Noether Normalization X will be "n-dimensional". We'll see that this is the case as soon as we define what dimension of an algebraic set is.*

As usual in Algebraic Geometry to prove Noether Normalization we need to translate the geometric statement into the language of algebra. Since Noether Normalization deals with finite maps $X \to \mathbb{A}^n$, the algebraic side will deal with integral extensions $k[x_1, \dots, x_n] \to k[X]$.

Definition 6.3. *Let k be a field and let S be an k-algebra. (Note that this is the same thing as an embedding of rings $k \subset S$). We say that elements $s_1, \dots, s_n$ of S are* algebraically independent *over k if the k-algebra homomorphism*

$$k[x_1, \dots, x_n] \to S$$

sending x_i to s_i for each i, is an injective homomorphism.

Equivalently, $s_1, \dots, s_n$ are algebraically independent if there are no nonzero n-variable polynomials with coefficients in k which vanish at $(s_1, \dots, s_n)$.

Example 6.4. *Let $R = k[x, y]$ be the polynomial ring. Then x, y are algebraically independent over k.*

Example 6.5. *Let $R = k[x, y]/(y^2 - x^3)$. The elements x, y are algebraically dependent over k, as they satisfy an equation $y^2 - x^3 = 0$.*

The next Theorem describes the structure of finitely generated algebras.

Theorem 6.6 (Noether Normalization Theorem, algebraic form)**.** *Let k be an infinite field, and R a finitely-generated k-algebra. Then R is an integral extension of a polynomial ring over k.*

More precisely, there are algebraically independent elements $s_1, \dots, s_n$ of S such that S is an integral extension of the subring $k[s_1, \dots, s_n]$:

$$k \subset k[s_1, \dots, s_n] \subset R.$$

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Proof. Let $s_1, \dots, s_n$ generate S as k-algebra. We do induction on n. When $n = 0$, $S = k$ and there is nothing to prove.

If $s_1, \dots, s_n$ are algebraically independent, then the homomorphism $k[x_1, \dots, x_n]$ sending x_i to s_i is an isomorphism, and again there is nothing to prove.

So assume that $n > 0$ and that the generators $s_1, \dots, s_n$ are algebraically dependent, and let $f(s_1, \dots, s_n) = 0$ be a relation between them. Here $f \in k[x_1, \dots, x_n]$ is a polynomial.

Consider first the case when the polynomial f has the form

$$f(x_1, \dots, x_n) = x_n^d + a_{n-1}(x_1, \dots, x_{n-1})x_n^{d-1} + \dots + a_1(x_1, \dots, x_{n-1})x_n + a_0(x_1, \dots, x_{n-1}) \tag{6.1}$$

that is it's monic considered as a polynomial in x_n. In this case since $f(s_1, \dots, s_n) = s_n^d + \dots = 0$, we see that s_n is integral over the subalgebra S' generated by $s_1, \dots, s_{n-1}$ and we apply the induction hypothesis to S': S' will be integral over a polynomial ring, and since being integral is a transitive relation, same applies to S.

It remains to show that after a change of coordinates we will have (6.1). Let $\deg(f) = d$ and let f_d be the sum of degree d monomials of f so that

$$f(x_1, \dots, x_n) = f_d(x_1, \dots, x_n) + \text{ lower degree monomials.}$$

Now assume that x_n is one of the variables which enters f_d nontrivially, otherwise renumber the variables.

Now apply the change of coordinates $x_i' = x_i - a_i x_n$, for $i < n$ and see how the polynomial f changes. We write the new polynomial as a polynomial in x_n:

$$\begin{aligned} f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1}x_n, x_n) &= f_d(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1}x_n, x_n) + \text{ lower degree monomials} \\ &= f_d(a_1, \dots, a_{n-1}, 1)x_n^d + \text{ lower degree terms in } x_n. \end{aligned}$$

Thus if we pick general $a_1, \dots, a_{n-1} \in k$, then the coefficient $f_d(a_1, \dots, a_{n-1}, 1) \in k$ is nonzero (here we use that k is infinite), and after dividing by this coefficient the equation will take the form (6.1). □

Proof of the geometric form of Noether Normalization. We apply the algebraic form of Noether Normalization to finitely generated k-algebra $S = k[X]$. Then the k-algebra embedding $k[x_1, \dots, x_n] \subset S$ translates into a polynomial map $\phi : X \to \mathbb{A}^n$. This polynomial map is finite since the corresponding extension of k-algebras is integral.

Proving that ϕ is surjective requires a bit more technology, and I omit the proof of this step (see Atiyah-Macdonald, Theorem 5.10). □