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CHAPTER 8: DIMENSION THEORY

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Among the basic concepts associated to geometric objects there is a notion of dimension. One spends a lot of time introducing dimension in linear algebra, which then is generalized in differential geometry or complex geometry. There is even dimension of fractals: not an integer, but rather a real number in general!

For an algebraic set X we define its dimension, also called Krull dimension in terms of chains of irreducible algebraic subsets

$$X_0 \subset X_1 \subset \cdots \subset X,$$

thus generalizing the dimension defined for vector spaces.

When developing the geometric concept in algebraic geometry we usually start on the level of commutative algebra. This is also the case with the Krull dimension.

8.1. **Heights of ideals and Krull dimension of rings.**

Definition 8.1. *Let R be a commutative ring. The* **height** *of a prime ideal $P \subset R$, denoted $\operatorname{ht} P$, is the length h of the longest chain of prime ideals*

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_h = P.$$

(Note that by the length here we mean the number of inclusions, that is the number of prime ideals plus one.) If there are arbitrarily long such chains we set $\operatorname{ht} P = \infty$.

Remark 8.2. *In Noetherian rings, ideals have finite height.*

Example 8.3. *Let $R = \mathbb{C}[x]$. Maximal ideals correspond to linear polynomials $x - a$, and 0 is non-maximal prime ideal, thus we have chains of the type $0 \subset (x - a)$, and no other chains. We see that*

$$\operatorname{ht}(0) = 0$$
$$\operatorname{ht}(x - a) = 1.$$

Example 8.4. *Let $R = \mathbb{C}[x, y]$. Maximal ideals correspond to points $(a_1, a_2) \in \mathbb{C}^2$ and we have chains*

$$(0) \subset (x - a_1) \subset (x - a_1, y - a_2).$$

We see that $\operatorname{ht}(m_a) \geq 2$. We'll see later that in fact $\operatorname{ht}(m_a) = 2$, but this is harder to prove, since we need to bound all possible chains.

Definition 8.5. *The (Krull) dimension $\dim(R)$ of a ring R in general is the maximum of the lengths of all chains of primes, and infinity if the lengths are unbounded.*

Example 8.6. *The ring $\mathbb{C}$ has only one prime ideal (0) so that $\dim(\mathbb{C}) = 0$. The Krull dimension should not be confused with the dimension of a vector space!*

Example 8.7. *The study of heights in Example 8.3 implies that $\dim(\mathbb{C}[x]) = 1$.*

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Remark 8.8. *Dimensions of Noetherian rings may be infinite! The simplest example is a bit complicated: it is a certain localization of a polynomial ring in infinitely many variables, where for every n there will be an ideal I_n of height $\mathrm{ht}(I_n) \geq n$.*

Lemma 8.9. *Let R be a ring and consider $\overline{R} = R/Nil(R)$, the quotient by the Nilradical. Then $\dim(R) = \dim(\overline{R})$.*

Proof. Every prime ideal in R contains $Nil(R)$: indeed one of the characterizations of $Nil(R)$ was the intersection of all prime ideals. Thus we have a bijection $\mathrm{Spec}(R) = \mathrm{Spec}(\overline{R})$, which is obviously order-preserving, hence $\dim(R) = \dim(\overline{R})$. ☐

Example 8.10. *Let $R = \mathbb{C}[x]/(x^n)$. We have $Nil(R) = (x)$. Using the previous Lemma we have*

$$\dim(R) = \dim(\overline{R}) = \dim(\mathbb{C}) = 0.$$

Furthermore in the homework assignment you will prove that every Artinian ring has dimension zero. This is one reason why I refer to Artinian rings as "small".

Lemma 8.11. *Let R be a local ring with maximal ideal m. Then*

$$\dim(R) = \mathrm{ht}(m).$$

Proof. We can put m on top of any chain of prime ideals, so that every maximal chain of prime ideals will end with m. ☐

Lemma 8.12. *Let $P \subset R$ be an ideal, and let R_P denote the localization at P (as usual). Recall that R_P is a local ring with maximal ideal m_P. Then*

$$\mathrm{ht}(P) = \mathrm{ht}(m_P) = \dim(R_P).$$

Proof. There is an order-preserving bijection between prime ideals in R_P and prime ideals in R which are contained in P. This gives a bijection on chains, which leads to the equality of heights. ☐

Example 8.13. *Let $R = \mathbb{Z}$. Maximal ideals are (p) for primes p, and the only non-maximal prime ideal is (0). We have chains $(0) \subset (p)$ and no other chains. This shows that $\mathrm{ht}(0) = 0$ and that $\mathrm{ht}(p) = 1$.*

Now we may consider localizations:

- *$P = (0)$, then $R_P = \mathbb{Q}$ (we invert all non-zero elements), and $\dim(\mathbb{Q}) = 0 = \mathrm{ht}(0)$*
- *$P = (p)$, then $R_P = \mathbb{Z}_{(p)}$ (we invert all primes other than p), and $\dim(\mathbb{Z}_{(p)}) = 1 = \mathrm{ht}(p)$.*

Example 8.14. *Consider the localization $R = \mathbb{C}[x]_{(x-a)}$, $a \in \mathbb{C}$. When defining localization I explained that its geometric meaning is to encapture information about a neighbourhood of a point. For dimension we have:*

$$\dim \mathbb{C}[x]_{(x-a)} = \mathrm{ht}_{\mathbb{C}[x]}(x-a) = 1,$$

thus the ring $R = \mathbb{C}[x]_{(x-a)}$ "knows" that it is a localization of one-dimensional object $\mathbb{A}^1$.

8.2. **The dimension of algebraic sets.**

Definition 8.15. *The* **dimension** *of an algebraic set $X \subseteq \mathbb{A}^n$ is the dimension of its coordinate ring $k[X]$.*

Theorem 8.16. *Let $X \subseteq \mathbb{A}^n$ be an algebraic set, let $I = I(X) \subseteq k[x_1, \ldots, x_n]$.*

The following numbers are equal:

(a) *The dimension of* X
(b) *The maximal length of a chain of prime ideals in* $k[x_1, \ldots, x_n]$ *containing* I
(c) *The maximal length of a chain of irreducible algebraic sets contained in* X

Proof. One uses bijections between the following sets coming from the Hilbert Nullstellensatz:

$$\{\text{Prime ideals in } k[X]\},$$
$$\{\text{Prime ideals in } k[x_1, \ldots, x_n] \text{ which contain } I(X)\},$$
$$\{\text{Irreducible algebraic subsets } Z \subset X\}.$$

□

Lemma 8.17. *If* $X \subseteq Y$ *are algebraic sets with* Y *irreducible, and* $\dim(X) = \dim(Y)$*, then* $X = Y$*.*

Proof. If $X \neq Y$, then any chain of irreducible subsets of X can be extended by adding Y at the end (since Y is irreducible!), in which case maximal chains in X and Y can never have same length. □

Example 8.18. *If* Y *is reducible the Lemma above does not apply: let* $X = V(xy) \subset \mathbb{A}^1$*, and* $Y = V(x) \subset \mathbb{A}^1$*. Then* X *consists of two irreducible components* $Y = V(x)$ *and* $Y' = V(y)$*, and* $\dim(X) = \dim(Y) = 1$.

Proposition 8.19. *Let* X *be algebraic set with irreducible components* $Z_1, \ldots, Z_n$*. Then*

$$\dim(X) = \max(\dim(Z_1), \ldots, \dim(Z_n)).$$

Proof. Let us use the description of dimension as a maximal length of a chain of irreducible algebraic sets contained in X:

$$X_0 \subsetneq \cdots \subsetneq X_h \subset X.$$

It is obvious that $\dim(Z_j) \leq \dim(X)$ (every chain in Z_j is also a chain in X) so that $\max(\dim(Z_1), \ldots, \dim(Z_n)) \leq \dim(X)$. By definition irreducible components of X are maximal irreducible subsets in X, hence $X_h = Z_j$ for some j, this means that in fact $\dim(X) = \dim(Z_j)$, and we are done. □

Example 8.20. *Let* $X = V(zx, zy) \subset k^3$*. When solving the system of equations* $zx = 0, zy = 0$ *we get two cases: either* $z = 0$ *or* $x = y = 0$*. This means that we have*

$$X = V(z) \cup V(x, y),$$

a union of plane isomorphic to $\mathbb{A}^2$ *and a line isomorphic to* $\mathbb{A}^1$*. According to the Proposition,* $\dim(X) = 2$ *(assuming that* $\dim(\mathbb{A}^2) = 2$*, see Theorem 8.25 below).*

8.3. **Finite polynomial maps and dimension.**

Theorem 8.21. *Let* $\phi : X \to Y$ *be a finite surjective polynomial map of algebraic sets. Then* $\dim(X) = \dim(Y)$*.*

Note that we do not require k to be algebraically closed. The theorem follows from a more general result when one takes $R = k[Y]$ and $S = k[X]$:

Theorem 8.22. *Let* $R \subset S$ *be a integral extension. Then* $\dim(R) = \dim(S)$*.*

Now this theorem relies on the following algebraic result which relates primes in integral extensions.

Theorem 8.23 (Going up Theorem). *Let $R \subset S$ be an integral extension.*

(a) *For any prime $P \subset R$ there exists a prime $Q \subset S$ with $P = Q \cap R$.*
(b) *If $Q \subseteq Q' \subset S$ are primes with $Q \cap R = Q' \cap R$, then $Q = Q'$.*
(c) *For any chain of primes $P_0 \subset \cdots \subset P_n$ of R and prime $Q_0 \subset S$ such that $Q_0 \cap R = P_0$ there exist primes $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ in S with $P_i = Q_i \cap R$.*

Proof of Theorem 8.22. To show that $\dim(Y) \geq \dim(X)$ we take a finite strict chain of primes $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_h$ in R and use Going Up (c) to lift this to a chain $Q_0 \subset Q_1 \subset \cdots \subset Q_h$ of primes in S. This new chain is obviously strict (otherwise, if $Q_i = Q_{i+1}$, then $P_i = P_{i+1}$), and this shows $\dim(Y) \geq \dim(X)$.

To show the converse inequality $\dim(X) \geq \dim(Y)$ we take a strict chain of primes $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_h$ in S and intersect these primes with R giving a chain $P_0 \subset Q_1 \subset \cdots \subset P_h$ of primes in R; now this chain is also strict by Going Up (b) and this shows $\dim(X) \geq \dim(Y)$. □

Proof of the Going Up Theorem. Proof of part (a):

- Consider $U = R \setminus P$, and replace R and S by localizations $R_P = U^{-1}R$ and $U^{-1}S$ respectively. Then $R_P \to U^{-1}S$ is injective (localization is exact!) and integral, because every element in $U^{-1}S$ has the form $\frac{s}{u} = s \cdot \frac{1}{u}$ with $s \in S$ integral and $\frac{1}{u} \in R_P$ is unit, so $\frac{s}{u}$ is integral.
- Thus we may assume that R is local with maximal ideal P
- Now any prime ideal $Q \subset S$ which contains PS will satisfy $Q \cap R = P$ because $Q \cap R \supset P$ and P is maximal.
- Thus we only need to show that PS is a proper ideal, that is $PS \neq S$.
- We use Nakayama's Lemma applied to finitely generated R-module S (S is a integral over R, hence it is a finitely generated R-module): $PS = S \implies S = 0$, which is impossible

Proof of part (b):

- We may replace R and S with R/P and S/Q. Indeed note $R/P \to S/Q$ is still injective, so that we have a ring extension $R/P \subset S/Q$ and it is obviously integral. We thus may assume that R, S are domains and that $P = 0$, $Q = 0$ and $0 \subseteq Q' \subset S$.
- Let us show that $Q' \cap R = 0$ is impossible unless $Q' = 0$
- Take an element $s \in Q'$ and write its equation as $s^d + r_{d-1}s^{d-1} + \cdots + r_0 = 0$.
- We may assume $r_0 \neq 0$, otherwise divide the equation by s (we are in a domain!)
- Hence we may rewrite the equation $r_0 = s \cdot (\dots) \in Q' \cap R = 0$, a contradiction!

Proof of part (c):

- Using induction on n we can reduce to the case $n = 1$.
- S/Q_0 is integral extension of R/P_0
- Going Up (a) shows that there exists a prime Q_1/Q_0 of S/Q_0 lying over P_1/P_0.

□

Example 8.24. *To illustrate the power of Theorem 8.22 we look at the following familiar examples of integral extensions:*

(1) $\dim \mathbb{R} = \dim \mathbb{C} = 0$
(2) $\dim k[x]/(x^2) = \dim k = 0$
(3) $\dim \mathbb{Z}[\sqrt{2}] = \dim \mathbb{Z} = 1$
(4) $\dim k[x, y]/(y^2 - x^3) = \dim k[x] = 1$*; this tells us that* $y^2 - x^3 = 0$ *is a curve!*
(5) $\dim k[x, y]/(z^2 - xy) = \dim k[x, y] = 2$*; this tells us that* $z^2 - xy = 0$ *is a surface!*

We are ready to fulfill our expectation about dimension of the affine space.

Theorem 8.25. $\dim(\mathbb{A}^n) = n$.

Proof. We do a proof by induction on n. The cases $n = 0, 1$ already have been considered earlier.

A chain
$$0 \subsetneq \mathbb{A}^1 \subsetneq \cdots \subsetneq \mathbb{A}^n$$
of irreducible algebraic subsets has length n, so $\dim(\mathbb{A}^n) \geq n$.

We need to show that there is no chain of greater length. Consider a maximal chain of irreducible algebraic subsets:
$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{h-1} \subsetneq Z_h = \mathbb{A}^n.$$

Let $P = I(Z_{h-1}) \subset k[x_1, \ldots, x_n]$ be the ideal. Since Z_{h-1} is irreducible P is prime. Let $f \in P$ be an arbitrary non-zero element. Since P is prime, we may assume f to be irreducible (since one of the factors always will be in P). This translates to a chain
$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{h-1} \subset V(f) \subset Z_h = \mathbb{A}^n.$$
Since the original chain was assumed to be maximal, one of the new inclusions is not proper. If we had $V(f) = \mathbb{A}^n$, this would mean that $f \in I(V(f)) = I(\mathbb{A}^n) = 0$, which contradicts to our assumption $f \neq 0$. Thus we have $V(f) = Z_{h-1}$.

Let us show that $\dim(V(f)) \leq n - 1$, this will imply that $h - 1 \leq n - 1$ and $h \leq n$, so we will be able to bound the original chain.

Now to bound $\dim(V(f))$ we use Noether Normalization: there is a finite surjective polynomial map $\phi : V(f) \to \mathbb{A}^k$ and from the proof of Noether normalization it follows[1] that $k = n - 1$.

By induction hypothesis we have $\dim \mathbb{A}^{n-1} = n - 1$, so by Theorem 8.21 $\dim(V(f)) = n - 1$ as well. This means that chains of irreducible subsets of $\dim(V(f))$ have lengths bounded by $n - 1$ so that $h \leq n$, and we are done. □

[1]Indeeed, $k[V(f)] = k[x_1, \ldots, x_n]/(f)$ is generated by $x_1, \ldots, x_n$, and since $x_1, \ldots, x_n$ are algebraically dependent the first step of the inductive proof of Noether normalization will present $k[V(f)]$ as an integral extension of a polynomial ring $k[y_1, \ldots, y_{n-1}]$.