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CHAPTER 9: TENSOR PRODUCTS

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Given a ring R and two R-modules M and N we will define their tensor product $M \otimes_R N$ and investigate its properties. We then discuss the geometric meaning of the construction: if $R = k$ is a field, $M = k[X]$ and $N = k[Y]$ are coordinate rings of algebraic sets X and Y, then $k[X] \otimes_k k[Y]$ is a k-algebra isomorphic to the coordinate ring of $X \times Y$.

9.1. **Tensor products of modules.**

Definition 9.1. *Let M, N, K be R-modules. A map $f : M \times N \to K$ is called R-bilinear if f is an homomorphism of R-modules in both variables, that is if*

$$f(rm_1 + m_2, n) = rf(m_1, n) + f(m_2, n), \text{ for all } r \in R,\ m_1, m_2 \in M,\ n \in N$$

and

$$f(m, rn_1 + n_2) = rf(m, n_1) + f(m, n_2), \text{ for all } r \in R,\ n_1, n_2 \in N,\ m \in M.$$

Let M, N be two R-modules. Let $F(M, N)$ be the free R-module with basis given by symbols $m \otimes n$ for $m \in M$, $n \in N$:

$$F(M, N) = \bigoplus_{m \in M, n \in N} R \cdot m \otimes n.$$

The R-module is very large, e.g. if M or N is uncountable as a set, then $F(M, N)$ has an uncountable basis.

Definition 9.2. *The tensor product $M \otimes_R N$ is the R-module defined as a quotient*

$$F(M, N)/B(M, N)$$

with $B(M, N)$ submodule generated by relations:

$$\begin{aligned} (m_1 + rm_2) \otimes n - m_1 \otimes n - rm_2 \otimes n, \\ m \otimes (n_1 + rn_2) - m \otimes n_1 - rm \otimes n_2. \end{aligned} \tag{9.1}$$

Thus elements of $M \otimes_R N$ have the form

$$\sum_{i=1}^{k} m_i \otimes n_i$$

for $m_i \in M$, $n_i \in N$, and there are relations

$$\begin{aligned} (m_1 + rm_2) \otimes n = m_1 \otimes n + rm_2 \otimes n, \\ m \otimes (n_1 + rn_2) = m \otimes n_1 + rm \otimes n_2, \end{aligned} \tag{9.2}$$

for example

$$(2m) \otimes n = m \otimes n + m \otimes n = m \otimes (2n).$$

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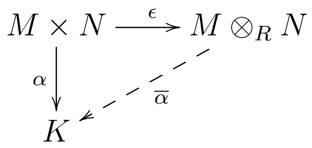
Elements of the kind $m \otimes n$ are called decomposable tensors. By construction decomposable tensors generate $M \otimes_R N$. In general there is no simple way to tell whether two elements of a tensor product are equal.

If the base ring R is fixed, we simply write $M \otimes N$ for $M \otimes_R N$. The definition of a tensor product $M \otimes_R N$ reminds the definition of a bilinear map $M \times N \to K$. The following Theorem tells that in fact the tensor product is "the smallest" R-module which receives an R-bilinear map from $M \times N$.

Theorem 9.3. *There is an R-bilinear map $\epsilon : M \times N \to M \otimes_R N$ defined as $\epsilon(m, n) = m \otimes n$. Furthermore, for any R-bilinear map $\alpha : M \times N \to K$ there is a unique R-module homomorphism $\overline{\alpha} : M \otimes_R N$ satisfying $\overline{\alpha} \circ \epsilon = \alpha$:*



Thus maybe we can't really say what elements of $M \otimes_R N$ are, but the Theorem gives us a way to construct R-module homomorphisms $M \otimes_R N \to K$ for any R-module K.

Proof. It follows from relations defining the tensor product $M \otimes_R N$ that ϵ is bilinear.

Given a bilinear map $\alpha : M \times N \to K$ we first define an R-module homomorphism

$$\widetilde{\alpha} : F(M \times N) \to K$$

Since $F(M \times N)$ is a free R-module we may send basis elements anywhere we want, so we define $\widetilde{\alpha}(m \otimes n) = \alpha(m, n)$.

Now because α is bilinear, the submodule (9.1) of relations $B(M, N) \subset F(M, N)$ is annihilated by $\widetilde{\alpha}$:

$$\widetilde{\alpha}\left((m_1 + rm_2) \otimes n - m_1 \otimes n + rm_2 \otimes n\right) = \alpha(m_1 + rm_2, n) - \alpha(m_1, n) - r\alpha(m_2, n) = 0$$
$$\widetilde{\alpha}\left(m \otimes (n_1 + rn_2) - m \otimes n_1 + rm \otimes n_2\right) = \alpha(m, n_1 + rn_2) - \alpha(m, n_1) - r\alpha(m, n_2) = 0.$$

It follows that $\widetilde{\alpha}$ descends to give a R-homomorphism $\overline{\alpha} : M \otimes_R N \to K$ defined as

$$\overline{\alpha}(\sum_{i=1}^{k} m_i \otimes n_i) = \sum_{i=1}^{k} \alpha(m_i, n_i).$$

We see that $\overline{\alpha} \circ \epsilon = \alpha$.

Uniqueness of $\overline{\alpha}$ follows the fact that its values on decomposable tensors $m \otimes n$ is determined by α, and decomposable tensors span $M \otimes_R N$. □

Proposition 9.4. *If $R = k$, a field, and V and W are k-vector spaces with bases $V = \langle e_i \rangle_{i \in I}$, $W = \langle f_j \rangle_{j \in J}$, then $V \otimes_k W$ is a k-vector space with basis $e_i \otimes f_j$, $i \in I$, $j \in J$.*

We see that for vector spaces $\dim_k(V \otimes W) = \dim_k(V) \cdot \dim_k(W)$ in the case dimensions of V and W are finite.

Proof. By construction the elements $e_i \otimes f_j$ generate the k-vector space $V \otimes W$. We need to show that these elements are linearly independent.

For every $i \in I$, $j \in J$ consider k-bilinear maps

$$\pi_{i,j} : V \times W \to k$$
$$\left(\sum_i a_i e_i, \sum_j b_j f_j\right) \mapsto a_i b_j.$$

Using Theorem 9.3 these k-bilinear maps induce k-module homomorphisms $\overline{\pi}_{i,j}$ satisfying:

$$\overline{\pi}_{i,j}\left((\sum_i a_i e_i) \otimes (\sum_j b_j f_j)\right) = a_i b_j \in k, \tag{9.3}$$

and

$$\overline{\pi}_{i,j}\left(\sum_{i,j} a_{i,j} e_i \otimes f_j\right) = a_{ij} \in k.$$

Existence of such coordinate homomorphisms implies linear independence of the spanning set:

$$\sum_{i,j} a_{ij} e_i \otimes f_j = 0 \implies a_{ij} = \overline{\pi}_{i,j}(\sum_{i,j} a_{ij} e_i \otimes f_j) = 0 \text{ for all } i, j.$$

□

Example 9.5. *Let $R = k$, a field, and let $M = k[x]$, $N = k[y]$, polynomial rings in one variable considered as k-vector spaces. Applying the Proposition above to the bases $k[x] = \langle 1, x, x^2, \dots \rangle$, and $k[y] = \langle 1, y, y^2, \dots \rangle$ we obtain a k-basis for $k[x] \otimes_k k[y]$:*

$$x^i \otimes y^j, \ i \geq 0, j \geq 0.$$

Thus as k-vector space $k[x] \otimes_k k[y]$ is isomorphic to $k[x, y]$. We'll see later that this is in fact an isomorphisms of k-algebras (at the moment tensor product does not have an algebra structure).

Example 9.6. *Note how tensor product of modules depends on the base ring: $k[x] \otimes_k k[x] \simeq k[x_1, x_2]$ but $k[x] \otimes_{k[x]} k[x] \simeq k[x]$, see (1) in Proposition 9.7 below. Tensoring over bigger ring $k[x]$ rather than over k gives more relations among the tensors, and yields a smaller module. For instance $x \otimes 1$ and $1 \otimes x$ are distinct elements in $k[x] \otimes_k k[x]$, but coincide in $k[x] \otimes_{k[x]} k[x]$.*

Proposition 9.7. *We have the following natural isomorphisms of R-modules:*

1. $R \otimes_R M \simeq M$
2. $M \otimes_R N \simeq N \otimes_R M$
3. $(M_1 \otimes_R M_2) \otimes_R M_3 \simeq M_1 \otimes_R (M_2 \otimes_R M_3)$
4. $(M_1 \oplus M_2) \otimes_R N \simeq (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$.

Proof. All these properties are proved in the same way: one constructs R-module homomorphisms in both directions using Theorem 9.3 and then checks that both compositions are identities. Since decomposable tensors generate tensor products it suffices to check whether compositions are identities on decomposable tensors.

I prove (1) while (2), (3) and (4) are done similarly. Let us construct R-module homomorphisms

$$\phi : R \otimes_R M \to M$$
$$\psi : M \to R \otimes_R M.$$

The first one is defined by the rule $\phi(r \otimes m) = rm$, this is well-defined by Theorem 9.3, since the RHS is bilinear in the two arguments. The second homomorphism is defined by $\psi(m) = 1 \otimes m$. This is well-defined since the RHS is linear in m.

We compute the two compositions:

$$\phi\psi(m) = \phi(1 \otimes m) = m$$
$$\psi\phi(r \otimes m) = \psi(rm) = 1 \otimes rm = r \otimes m,$$

these are identities, and we are done. □

Proposition 9.8. *If $f_1 : M_1 \to N_1$, $f_2 : M_2 \to N_2$ are R-module homomorphisms, then there is an R-module homomorphism $f_1 \otimes f_2 : M_1 \otimes M_2 \to N_1 \otimes N_2$ defined via $(f_1 \otimes f_2)(m_1 \otimes m_2) = f_1(m_1) \otimes f_2(m_2)$.*

In particular for every $f : M \to N$ and K there is an R-module homomorphism $f \otimes id : M \otimes K \to N \otimes K$.

Proof. One of the problems in the Homework assignment. □

9.2. **Extension of scalars and tensor products of algebras.**

Proposition 9.9. *Let R be a ring, M an R-module and S an R-algebra. Then the R-module $S \otimes_R M$ has a structure of S-module defined by*

$$t \cdot (\sum_{i=1}^{k} s_i \otimes m_i) := \sum_{i=1}^{k} (ts_i) \otimes m_i.$$

This construction is called extension of scalars: we make R-module M into an S-module in the most efficient way.

Proof. For every t we define a map m_t:

$$m_t : S \otimes_R M \to S \otimes_R M$$
$$s \otimes m \mapsto ts \otimes m$$

Because the RHS is bilinear in s and m this map gives a well-defined homomorphism of R-modules by Theorem 9.3. The axioms for $S \otimes_R M$ to be an S-module are encoded as:

$$m_1 = id$$
$$m_{t_1 t_2} = m_{t_1} \circ m_{t_2}$$
$$m_{t_1 + t_2} = m_{t_1} + m_{t_2},$$

and these follow immediately from the construction of m_t. □

Example 9.10. *We take $R = \mathbb{R}$, $M = \mathbb{R}[x]$, considered as $\mathbb{R}$-module, and $S = \mathbb{C}$, considered as $\mathbb{R}$-algebra. Let us consider the $\mathbb{C}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]$. As an $\mathbb{R}$-vector space $\mathbb{C}$ has a basis $1, i$, and $\mathbb{R}[x]$ has a basis $1, x, x^2, \dots$. Thus using Proposition 9.4 we find an $\mathbb{R}$-basis:*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x] = \langle 1 \otimes 1, i \otimes 1, 1 \otimes x, i \otimes x, 1 \otimes x^2, i \otimes x^2, \dots \rangle.$$

In other words elements of this module can be identified with

$$f(x) + ig(x),\ f(x), g(x) \in \mathbb{R}[x].$$

One checks that the $\mathbb{C}$-action is what it should be giving an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x] \simeq \mathbb{C}[x].$$

Proposition 9.11. *Let R be a ring and S and T be R-algebras. Then the R-module $S \otimes T$ has a structure of R-algebra with multiplication defined as*

$$\left(\sum_{i=1}^{k} s_i \otimes t_i\right) \cdot \left(\sum_{j=1}^{l} s'_j \otimes t'_j\right) = \sum_{i,j} s_i s'_j \otimes t_i t'_j.$$

Proof. The proof is similar to that of Proposition 9.9 but a bit more involved. □

Example 9.12. *Consider $S = k[x]$ and $T = k[y]$ as k-algebras. Let us consider $k[x] \otimes_k k[y]$ as a k-algebra. In Example 9.5 we constructed a k-module isomorphism*

$$k[x] \otimes_k k[y] \simeq k[x, y]$$
$$x^i \otimes x^j \mapsto x^i y^j$$

We check that this map is actually an isomorphism of k-algebras. For that we need to check that multiplication of the basis elements on both sides is respected by the map. This follows from how the ring structure is defined on the tensor product in Proposition 9.11:

$$(x^{i_1} \otimes y^{j_1}) \cdot (x^{i_2} \otimes y^{j_2}) = x^{i_1+i_2} \otimes y^{j_1+j_2}.$$

9.3. **Products of algebraic sets.**

Lemma 9.13. *Let $X \subset \mathbb{A}^n$ be algebraic subset with ideal $I(X) \subset k[x_1, \dots, x_n]$. Then $X \times \mathbb{A}^1 \subset \mathbb{A}^{n+1}$ is an algebraic subset with ideal $I(X)[y] \subset k[x_1, \dots, x_n, y]$, that is the ideal generated by $I(X)$ in $k[x_1, \dots, x_n, y]$.*

Proof. We start by noticing that the vanishing locus of $I(X)$ considered as a subset of $k[x_1, \dots, x_n, y]$ is $X \times \mathbb{A}^1$, as the there are no restrictions in the y direction. To find the ideal of $X \times \mathbb{A}^1$ we consider a polynomial

$$h(x_1, \dots, x_n, y) = \sum_{i=0}^{k} a_i(x_1, \dots, x_n) y^i \in k[x_1, \dots, x_n, y]$$

and notice that h vanishes at $X \times \mathbb{A}^n$ if and only if for all $(x_1, \dots, x_n) \in X$ the polynomial

$$h(x_1, \dots, x_n, y) \in k[y]$$

vanishes on $\mathbb{A}^1$, that is its coefficients $a_i(x_1, \dots, x_n)$ are all zero. It follows that $h \in I(X \times \mathbb{A}^1)$ if and only if $h \in I(X)[y]$. □

Theorem 9.14. *Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be algebraic sets with ideals $I(X) \subset k[\mathbb{A}^n]$, $I(Y) \subset k[\mathbb{A}^m]$. Then $X \times Y \subset \mathbb{A}^{n+m}$ is an algebraic set and $I(X \times Y) = I(X)k[\mathbb{A}^{n+m}] + I(Y)k[\mathbb{A}^{n+m}]$.*

Proof. Denote the coordinates on $\mathbb{A}^{n+m} = \mathbb{A}^n \times \mathbb{A}^m$ be $x_1, \dots, x_n, y_1, \dots, y_m$.

We note that $X \times Y$ is the intersection of two "cylinders":

$$X \times Y = X \times \mathbb{A}^m \cap \mathbb{A}^n \times Y.$$

Using Lemma 9.13 and easy induction we see that both of the two cylinders above are algebraic, and hence their intersection is also algebraic. For the ideal of $X \times Y$ we have

$$\begin{aligned} I(X \times Y) &= I(X \times \mathbb{A}^m) + I(\mathbb{A}^n \times Y) = \\ &= I(X)[y_1, \dots, y_m] + I(Y)[x_1, \dots, x_m] = \\ &= I(X)k[\mathbb{A}^{n+m}] + I(Y)k[\mathbb{A}^{n+m}]. \end{aligned}$$

□

Theorem 9.15. *The coordinate algebra of $X \times Y$ is isomorphic to $k[X] \otimes_k k[Y]$.*

Proof. Let $X \subset \mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$ and $Y \subset \mathbb{A}^m$ with coordinates $y_1, \ldots, y_m$. By Theorem 9.14 we have

$$k[X \times Y] = k[\mathbb{A}^{n+m}]/I(X \times Y) = k[\mathbb{A}^{n+m}]/(I(X)k[\mathbb{A}^{n+m}] + I(X)k[\mathbb{A}^{n+m}]).$$

The proof is finished using identification $k[\mathbb{A}^{n+m}] \simeq k[\mathbb{A}^n] \otimes k[\mathbb{A}^m]$ (Example 9.12) and Lemma 9.16 below:

$$\begin{aligned} k[\mathbb{A}^{n+m}]/(I(X)k[\mathbb{A}^{n+m}] + I(X)k[\mathbb{A}^{n+m}]) &\simeq (k[\mathbb{A}^n] \otimes k[\mathbb{A}^m])/(I(X) \otimes k[\mathbb{A}^m] + k[\mathbb{A}^n] \otimes I(Y)) \simeq \\ &\simeq k[\mathbb{A}^n]/I(X) \otimes k[\mathbb{A}^m]/I(Y) \\ &\simeq k[X] \otimes k[Y]. \end{aligned}$$

□

Lemma 9.16. *Let $I_1 \subset R_1$, $I_2 \subset R_2$ be ideals in k-algebras R_1, R_2. Then there is a natural isomorphism of k-algebras*

$$R_1/I_1 \otimes_k R_2/I_2 \simeq (R_1 \otimes_k R_2)/(I_1 \otimes R_2 + R_1 \otimes I_2).$$

Proof. We construct k-algebra homomorphisms

$$\begin{aligned} R_1/I_1 \otimes_k R_2/I_2 &\to (R_1 \otimes_k R_2)/(I_1 \otimes R_2 + R_1 \otimes I_2) \\ \overline{r_1} \otimes \overline{r_2} &\mapsto \overline{r_1 \otimes r_2} \end{aligned}$$

and

$$\begin{aligned} (R_1 \otimes_k R_2)/(I_1 \otimes R_2 + R_1 \otimes I_2) &\to R_1/I_1 \otimes_k R_2/I_2 \\ \overline{r_1 \otimes r_2} &\mapsto \overline{r_1} \otimes \overline{r_2}. \end{aligned}$$

It is easy to see that these are well-defined and mutually inverse. □

Proposition 9.17. *Dimension of $X \times Y$ is $\dim(X) + \dim(Y)$.*

Proof. We apply Noether Normalization to X and Y to get finite surjective polynomial maps $\phi : X \to \mathbb{A}^n$, $\psi : Y \to \mathbb{A}^m$. Consider the map on the product:

$$\begin{aligned} \phi \times \psi : X \times Y &\to \mathbb{A}^{n+m} \\ (x, y) &\mapsto (\phi(x), \psi(y)). \end{aligned}$$

It is not hard to check that this polynomial map is also finite and surjective, so that

$$\dim(X \times Y) = n + m = \dim(X) + \dim(Y).$$

□

Remark 9.18. *Zariski Cancellation Problem asks about the following: if X is affine n-dimensional variety such that $X \times \mathbb{A}^1$ is isomorphic to $\mathbb{A}^{n+1}$, is it true that X is isomorphic to $\mathbb{A}^n$? This is not known for $n \geq 3$, and known to be false if characteristic of the base field k is positive.*