MAS439/MAS6320
CHAPTER 2: NOETHERIAN AND ARTINIAN MODULES

EVGENY SHINDER (IN PART BASED ON NOTES BY JAMES CRANCH)

2.1. **Chain conditions.** There are a lot of unpleasant rings out there, and a lot of nasty unpleasant modules over them. It's good to have notions of "nice", particularly if we want to do algebraic geometry.

Here are some straightforward ones, which come up repeatedly:

Definition 2.1. *Let R be a commutative ring, and let M be an R-module.*

- *We say that M satisfies the* ascending chain condition*, or that M is* Noetherian *if any ascending chain of submodules of M stabilises. That is, if we have R-modules*

$$N_1 \subset N_2 \subset N_3 \subset \cdots \subset M,$$

  then there is some n such that $N_n = N_{n+1} = N_{n+2} = \cdots$.
- *We say that M satisfies the* descending chain condition*, or that M is* Artinian *if any descending chain of submodules of M stabilises. That is, if we have R-modules*

$$M \supset N_1 \supset N_2 \supset N_3 \supset \cdots,$$

  then there is some n such that $N_n = N_{n+1} = N_{n+2} = \cdots$.

These are actually used more commonly as niceness conditions for rings rather than modules:

Definition 2.2. *Let R be a commutative ring.*

- *We say that R is a* Noetherian ring *if R is a Noetherian R-module. Since submodules of R are the same thing as ideals, this means that every increasing chain of ideals*

$$I_1 \subset I_2 \subset \cdots$$

  stabilises.
- *We say that R is a* Artinian ring *if R is an Artinian R-module. Similarly, this means that every decreasing chain of ideals*

$$I_1 \supset I_2 \supset \cdots$$

  stabilises.

Here are some examples of modules which are or are not Noetherian, and which are or are not Artinian.

Example 2.3. *Let $R = \mathbb{Z}$. Recall that R-modules are simply abelian groups.*

(1) *$\mathbb{Z}/m$ is Artinian and Noetherian as $\mathbb{Z}$-module. Indeed, since this module is a finite set, it only has finitely many submodules, and any chain stabilizes.*

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(2) *$\mathbb{Z}$ is Noetherian but not Artinian $\mathbb{Z}$-module. Decoding the terminology a bit, this says that every ascending chain of ideals in $\mathbb{Z}$ stabilises, but not every descending chain.*

Since $\mathbb{Z}$ is a principal ideal domain, any ideal is generated by some single element, and containment is the same as divisibility. Hence an ascending chain of ideals

$$(n_1) \subseteq (n_2) \subseteq (n_3) \subseteq \cdots$$

gives us a chain of nonnegative integers, each a factor of the one before:

$$\cdots \mid n_3 \mid n_2 \mid n_1.$$

That means that $n_1 \geq n_2 \geq n_3 \geq \cdots \geq 0$, so certainly the chain stabilises, and hence $\mathbb{Z}$ is a Noetherian $\mathbb{Z}$-module.

However, it is not Artinian. Consider the following descending chain of ideals:

$$(1) \supseteq (2) \supseteq (4) \supseteq (8) \supseteq (16) \supseteq \cdots .$$

This does not stabilise.

(3) *The set $\mathbb{C}^\times$ of nonzero complex numbers is an abelian group, and hence a $\mathbb{Z}$-module. Let p be a prime, and consider the subset*

$$U = \left\{x \in \mathbb{C}^\times \mid x^{p^n} = 1 \quad \text{for some } n\right\}.$$

That is a subgroup of $\mathbb{C}^\times$, and it is Artinian but not Noetherian as a $\mathbb{Z}$-module.

Let U_n be the subgroup consisting of all p^n-th roots of unity: that is, all $x \in \mathbb{C}^\times$ such that $x^{p^n} = 1$.

Then we have an ascending chain

$$U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \cdots$$

of submodules of U, which does not stabilise as they're all different. So U is not a Noetherian $\mathbb{Z}$-module.

However, it's not difficult to show (exercise!) that the U_i's are the only submodules of U. Hence any descending chain stabilises.

If we take modules which are "infinite-dimensional" in some appropriate sense, it's quite likely that they will be neither:

Example 2.4. *The set of complex polynomials $\mathbb{C}[x]$ is a complex vector space and hence a $\mathbb{C}$-module. In this example we show that it is neither Artinian nor Noetherian as a $\mathbb{C}$-module.*

Let U_n be the vector space of polynomials of degree at most n. Being a vector space, it is a submodule of $\mathbb{C}[x]$. Then the chain

$$U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \cdots$$

is an ascending chain which does not stabilise, showing that $\mathbb{C}[x]$ is not a Noetherian $\mathbb{C}$-module.

Similarly, let V_n be the space of polynomials which are a multiple of x^n (equivalently, those which have a root at 0 of order n). Then the chain

$$V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq \cdots$$

is a descending chain which does not stabilise, showing that $\mathbb{C}[x]$ is not an Artinian $\mathbb{C}$-module.

The following may not quite be obvious, and you should let it sink in.

Example 2.5. *The reader may think that the argument in the previous example says that* $\mathbb{C}[x]$ *is neither a Noetherian nor an Artinian ring. This is false.*

The proof there gives an ascending chain of submodules of $\mathbb{C}[x]$ regarded only as a $\mathbb{C}$-module, not a $\mathbb{C}[x]$-module. *This means that it does not show that* $\mathbb{C}[x]$ *is not Noetherian. In fact,* $\mathbb{C}[x]$ is *Noetherian, by a theorem of Hilbert.*

On the other hand, the example there of a descending chain is in fact a descending chain of $\mathbb{C}[x]$*-modules (they're all ideals in* $\mathbb{C}[x]$*). Hence* $\mathbb{C}[x]$ *is definitely not an Artinian ring.*

2.2. **Properties of Noetherian and Artinian modules.** We will see now that many of the properties of Noetherian and Artinian modules develop in the same way. To develop this theory we need to rely on quotient modules. So I start by explaining what are the **submodules of the quotient module**.

Proposition 2.6. *Let* $N \subset M$ *be an* R*-submodule, and let us consider the quotient module* M/N*. Then we have a natural bijection:*

$$\{\textit{Submodules } \overline{K} \subset M/N\} \leftrightarrow \{\textit{Intermediate submodules } N \subset K \subset M\} \tag{2.1}$$

Proof. If we have an intermediate submodule $N \subset K \subset M$, then we can define a submodule

$$\overline{K} := K/N \subset M/N.$$

as in the Third Isomorphism Theorem last week. In other words $\overline{K}$ consists of equivalence classes $k + N$, $k \in K$.

Conversely, given an arbitrary submodule $\overline{K} \subset M/N$ we can look at its preimage under the canonical quotient homomorphism $\phi : M \to M/N$:

$$K := \phi^{-1}(\overline{K}) = \{m \in N : m + N \in \overline{K}\}.$$

□

Lemma 2.7. *Let* $N \subset M$ *be a* R*-submodule. Then* M *is Noetherian (resp. Artinian) if and only if both* N *and* M/N *are Noetherian (resp. Artinian).*

Proof. We do the proof of the Noetherian property. The Artinian property is proved in the same way.

There are two statements to prove now. Let first us assume that M is Noetherian, and show that N and M/N are also Noetherian. A chain of submodules in N is at the same time a chain of submodules in M. Since the latter chains stabilize, N is Noetherian. Now take a chain of submodules in M/N. According to (2.1) this chain gives us a chain of submodules in M, and it must stablize, so must the original chain.

Now let us assume that both N and M/N are Noetherian. To show that M is Noetherian, consider a chain of submodules in M:

$$M_1 \subset M_2 \subset \cdots \subset M$$

and this gives rise to two other chains:

$$M_1 \cap N \subset M_2 \cap N \subset \cdots \subset N$$

and

$$(M_1 + N)/N \subset (M_2 + N)/N \subset \cdots \subset M/N.$$

Both of the new chains stabilize since N and M/N are Noetherian: for $i \geq n$ we have

$$M_i \cap N = M_{i+1} \cap N$$
$$(M_i + N)/N = (M_{i+1} + N)/N$$

Note that by from the Third Isomorphism Theorem we have a natural isomorphism

$$(M_i + N)/N \simeq M_i/(M_i \cap N).$$

Now comparing the two adjacent submodules M_i, M_{i+1} for $i \geq n$ we see that they have the same submodule

$$K := M_i \cap N = M_{i+1} \cap N$$

such that the quotient modules M_i/K, M_{i+1}/K are also the same as submodules in M/K. This forces $M_i = M_{i+1}$. □

Lemma 2.8. *If M and N are Noetherian (resp. Artinian) R-modules, then $M \oplus N$ is Noetherian (resp. Artinian) R-module.*

Proof. This follows very easily from Lemma 2.8. Consider an R-submodule

$$M = \{m, 0\} \subset M \oplus N.$$

Note the quotient is computed as

$$(M \oplus N)/M \simeq N$$

(for example, we can use First Isomorphism Theorem applied to the projection $M \oplus N \to N$).

The statement follows from Lemma 2.8. □

Theorem 2.9. *If R is a Noetherian (resp. Artinian) ring and M is a finitely generated R-module, then M is a Noetherian (resp. Artinian) R-module.*

Proof. Since M is finitely generated we know that M admits a surjective homomorphism

$$R^n \to M.$$

Using induction on Lemma 2.8 we see that R^n is Noetherian (resp. Artinian) R-module. The rest follows from Lemma 2.7. □

One thing that makes Noetherian modules especially nice is the following:

Theorem 2.10. *An R-module M is Noetherian if and only if every submodule $N \subset M$ is finitely generated.*

Proof. This generalizes the corresponding statement about rings (last semester): R is a Noetherian ring if and only if every ideal in R is finitely generated. The proof generalizes as well. □

2.3. **Properties of Noetherian and Artinian rings.** In contrast to the case of modules, properties of Noetherian and Artinian rings are very different.

The following theorem and its corollary demonstrate that Noetherian rings appear naturally in Algebraic Geometry.

Theorem 2.11 (Hilbert's Basis Theorem)**.** *If R is Noetherian then $R[x]$ is Noetherian.*

Proof. See last semester's notes. □

Corollary 2.12. *Let R be a quotient ring of a polynomial ring: $R = k[x_1, \dots, x_n]/I$, for some ideal I. Then R is a Noetherian ring.*

Proof. By the Hilbert Basis Theorem applied inductively we deduce that the ring $k[x_1, \ldots, x_n]$ is Noetherian. Now R is a finitely generated (actually, cyclic) $k[x_1, \ldots, x_n]$-module, hence Theorem 2.9 implies that R is Noetherian as $k[x_1, \ldots, x_n]$-module. Since ideals in R are the same as $k[x_1, \ldots, x_n]$ submodules this implies that R is Noetherian as a ring as well. □

Remark 2.13. *Of course $k[x_1, \ldots, x_n]$ is* **not Artinian** *ring, see Example 2.5.*

In fact Artinian rings are not at all as general as Noetherian rings, and their structure is much simpler.

Proposition 2.14. *An Artinian ring S which is an integral domain is a field.*

Proof. For every $0 \neq x \in S$ the following sequence of ideals in S:

$$\cdots \subset (x^{n+1}) \subset (x^n) \subset \cdots \subset (1) = S$$

stabilizes. This means that $yx^{n+1} = x^n$ for some $y \in S$. Since S is a domain we have

$$yx^{n+1} = x^n \implies yx = 1$$

so that x is invertible.

Since every nonzero element $x \in S$ is invertible, S is a field. □

Proposition 2.15. *If R is Artinian, then every prime ideal in R is maximal.*

Proof. Let $I \subset R$ be an ideal. Assume that I is prime so that the quotient ring $S = R/I$ is an integral domain. By Theorem 2.9 S is an Artinian ring and by the previous proposition S is a field. Thus I is a maximal ideal. □

Remark 2.16. *Surprisingly every Artinian ring is Noetherian (see Atiyah-MacDonald, Chapter 8) but we do not prove this fact.*