MAS439/MAS6320 - PROBLEMS - WEEK 3

Question 1. *Which of these rings are local? For the local ones specify the maximal ideal.*

- $\mathbb{Z}/p$ *(p prime)*
- $\mathbb{Z}/p \times \mathbb{Z}/q$ *(p,q primes)*
- $\mathbb{C}[x]/(x^5)$
- $\mathbb{C}[x, y]_{(y)}$

(4 marks)

The next two questions demonstrate the idea of checking properties of modules and homomorphisms *locally*, i.e. for every prime or maximal ideal. This idea is at the heart of Algebraic Geometry where maximal ideals form the set of points of an algebraic set.

Question 2. *Prove that the following conditions for an R-module M are equivalent:*

(a) $M = 0$
(b) $M_P = 0$ *for all prime ideals* $P \subset R$
(c) $M_m = 0$ *for all maximal ideals* $m \subset R$

[Hint: the implication (c) $\implies$ (a) is not very easy; you will need to use that every proper ideal in $I \subset R$ is contained in a maximal ideal.] (3 marks)

Question 3. *Use the result of the previous question to prove that the following conditions for an R-module homomorphism $f : M \to N$ are equivalent:*

(a) $f : M \to N$ *is an isomorphism*
(b) $f_P : M_P \to N_P$ *is an isomorphism for all prime ideals* P
(c) $f_P : M_m \to N_m$ *is an isomorphism for all maximal ideals* $m \subset P$

[Hint: even if you haven't completely solved Question 2, you can just apply its result as a black box here.] (3 marks)