MAS439/MAS6320 - PROBLEMS - WEEK 5

For a ring (or k-algebra) R the set of maximal ideals is called the *maximal spectrum of* R and is denoted by $\mathrm{Specm}(R)$, and the set of all prime ideals is called the *spectrum of* R and is denoted by $\mathrm{Spec}(R)$.

For any homomorphism of rings $f : R \to S$ the preimage of a prime ideal $P \subset S$ is prime, and we have a map going in the opposite direction for Spec:

$$\phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$$
$$P \subset S \mapsto f^{-1}(P) \subset R$$

(this is not necessarily the case for Specm).

Question 1. *Let R be an ring. For each of the homomorphisms $R \to S$ below show that the map* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is injective and describe the image:*

(a) *$S = R/I$ for an ideal $I \subset R$, and $R \to S$ is the quotient homomorphism*
(b) *$S = U^{-1}R$ for the multiplicative system $U = \{1, f, f^2, \dots\}$ for some element $f \in R$, and $R \to S$ is the localization homomorphism L_U*
(c) *$S = U^{-1}R$ for the multiplicative system $U = R \backslash P$ for some prime ideal $P \subset R$, and $R \to S$ is the localization homomorphism L_U*

(3 marks).

Recall that for the maximal spectrum of the polynomial ring over an algebraically closed field k the Hilbert Nullstellensatz gives a bijection:

$$K^n \leftrightarrow \mathrm{Specm}(k[x_1, \dots, x_n])$$
$$a \in K^n \mapsto I(a) = (x_1 - a_1, \dots, x_n - a_n)$$

(see page 38 of Tom's 2014/15 notes).

Question 2. *Use the bijection above to construct a natural bijection* $\mathrm{Specm}(k[X]) \leftrightarrow X$ *for any algebraic set $X \subset K^n$. (2 marks)*

Question 3. *Let k be an algebraically closed field. Explain which of the following polynomial maps of affine varieties are finite:*

(1) $\mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x^2, y^2)$
(2) $\mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$
(3) $\mathbb{A}^3 \to \mathbb{A}^2$, $(x, y, z) \mapsto (x, y)$
(4) $\mathbb{A}^2 \to \mathbb{A}^3$, $(x, y) \mapsto (x, y, 0)$
(5) $\mathbb{A}^1 \to C$, $t \mapsto (t^2, t^3)$, *where* $C = V(y^2 - x^3) \subset \mathbb{A}^2$

(5 marks)