MAS439/MAS6320 - PROBLEMS - WEEK 6

Question 1. *Let $R \subset S$ be an integral extension and assume that R and S are integral domains.*

(a) *Prove that if R is a field, then S is a field*

(b) *Prove that if S is a field, then R is a field*

[4 marks]

Question 2. *Recall that an algebraic set is called irreducible if $k[X]$ is a domain, and in this case the fraction field $k(X) = \mathrm{Frac}(k[X])$ is called the field of rational functions on X.*

Now let $\phi : X \to Y$ be a polynomial map of irreducible algebraic sets. Prove that the following conditions are equivalent:

(a) *The induced k-algebra homomorphism $\phi^* : k[Y] \to k[X]$ is injective.*

(b) *The image $\mathrm{Im}(\phi) \subset Y$ is dense in Y in Zariski topology, which by definition means that the smallest algebraic subset of Y which contains the image is Y itself.*

In this case we say that ϕ is **dominant**. *Show that every surjective polynomial map is dominant, and construct an example of a dominant but not surjective polynomial map. Finally show that if ϕ is dominant, then there is a well-defined field extension $k(Y) \subset k(X)$ induced by ϕ^*.*

[4 marks]

Question 3. *Let $\phi : X \to Y$ be a finite surjective polynomial map of irreducible algebraic sets. Prove that $k(X)$ is a finite-dimensional $k(Y)$-vector space (in this case we say that $k(X)$ is finite extension of $k(Y)$). We define the* **degree** *of ϕ as $\deg(\phi) := \dim_{k(Y)} k(X)$.*

Now think how the degree of ϕ compares with the size of the fibers of ϕ and try to prove your claim. [2 marks]