MAS439/MAS6320 - PROBLEMS - WEEK 7

Question 1. *Prove Lemma 7.9 from the notes. (4 marks)*

Question 2. *We call an integral domain R* **integrally closed** *if all elements of its fraction field* $\mathrm{Frac}(R)$ *which are integral over R belong to R. We call an irreducible algebraic set X* **normal** *if* $k[X]$ *is integrally closed.*

(a) *Show that $\mathbb{Z}$ is integrally closed, and explain that the same argument works to show that $k[x_1, \ldots, x_n]$ is integrally closed. This implies that $\mathbb{A}^n$ is normal.*

(b) *For an integral domain R define its* **integral closure** *S to be the set of elements of* $\mathrm{Frac}(R)$ *integral over R. Show that S is a ring and that S is integrally closed. When R and S are finitely generated k-algebras, the corresponding map of algebraic sets* $\mathrm{Specm}(S) \to \mathrm{Specm}(R)$ *is a finite map called* **normalization** *of* $X = \mathrm{Specm}(R)$.

(c) *Compute normalizations for curves $X_1 = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3\}$ and $X_2 = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + x^2\}$ and using these make a drawing how normalization can make algebraic sets "less singular".*

(6 marks)