DERIVED CATEGORIES: LECTURE 1

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References


1. Derived and triangulated categories

Let \( \mathcal{A} \) be an abelian category and let \( \text{Kom}(\mathcal{A}) \) denote the category of complexes of objects in \( \mathcal{A} \).

A morphism of complexes \( f : A^\bullet \to B^\bullet \) is called a quasi-isomorphism if the induced functors \( H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet) \) are isomorphisms for all \( i \in \mathbb{Z} \).

For example, if \( P^\bullet \to A \) is a resolution, then the morphism \( P^\bullet \to A[0] \) is a quasi-isomorphism.

In homological algebra we identify an object with all its resolutions, this leads naturally to considering all complexes, modulo quasi-isomorphism.

**Definition 1.1.** The derived category \( D(\mathcal{A}) \) of \( \mathcal{A} \) is obtained by formally inverting all quasi-isomorphisms.

\( D^+(\mathcal{A}) \) (resp. \( D^-(\mathcal{A}) \), resp. \( D^b(\mathcal{A}) \)) is defined as a full subcategory of \( D(\mathcal{A}) \) consisting of objects with cohomology bounded from below (resp. above, resp. above and below).

There is a fully faithful embedding of \( \mathcal{A} \) into \( D^b(\mathcal{A}) \) given by \( A \mapsto A[0] \). However, the derived category itself is not at all abelian. Derived category is a so-called triangulated category which means that it admits a shift functor \([1]\) and given a class of distinguished triangles.

The shift functor \([1]\) simply shifts the complex. Beware of the direction of cohomological grading shift: e.g. \( A[1] \) is \( A \) sitting in degree \(-1\)!

Instead of kernels and cokernels we have cones. If \( f : A^\bullet \to B^\bullet \) is a morphism of complexes, then the cone of \( f \) is defined as:

\[
\text{cone}(f)^p := B^p \oplus A^{p+1}
\]

and the differential maps \((b^p, a^{p+1}) \) to \((d(b^p) + f(a^{p+1}), -d(a^{p+1}))\).

The term cone comes from analogy with topology where for a continuous mapping \( f : X \to Y \) one defines a topological space \( C(f) \) as follows:

\[
C(f) := \frac{Y \sqcup (X \times [0, 1])}{f(x) \sim (y, 0), \ (y, 1) \sim (y', 1)}
\]
and this gives sequences such as

\[ X \to Y \to C(f) \to \Sigma X \]

which in turn give rise to long exact sequences of homology groups.

**Exercise 1.2.**

1. There is a short exact sequence of complexes

\[ 0 \to B^\bullet \to \text{cone}(f) \to A^\bullet \to 0. \]

2. If \( f : A^\bullet \to B^\bullet \) is a monomorphism, then \( \text{cone}(f) \) is homotopy equivalent to \( \text{coker}(f) \).

3. If \( f : A^\bullet \to B^\bullet \) is an epimorphism, then \( \text{cone}(f) \) is homotopy equivalent to \( \text{ker}(f)[1] \).

4. If \( f : A[0] \to B[0] \) is a morphism of complexes situated in degree 0, then

\[ \text{cone}(f) = [A \to B], \]

with \( B \) in degree 0 and with differential given by \( f \).

Sequences isomorphic to

\[ A^\bullet \to B^\bullet \to \text{cone}(f) \to A^\bullet[1] \]

are called distinguished triangles. One of the properties of distinguished triangles is that if

\[ A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1] \]

is distinguished, then

\[ B^\bullet \to C^\bullet \to A^\bullet[1] \to B^\bullet[1] \]

is also distinguished. We can visualise distinguished triangles like

\[
\begin{array}{ccc}
A^\bullet & \rightarrow^f & B^\bullet \\
[1] & \downarrow & \\
C^\bullet & \leftarrow & \end{array}
\]

Given a triangulated category \( C \) we produce long exact sequence as follows.

**Lemma 1.3.** Let \( C \) be a triangulated category. Then for any \( U \in C \) and any triangle

\[ A \to B \to C \to A[1] \]

the following sequences are exact:

\[ \cdots \to \text{Hom}(U, A) \to \text{Hom}(U, B) \to \text{Hom}(U, C) \to \text{Hom}(U, A[1]) \to \cdots \]

\[ \cdots \to \text{Hom}(C, U) \to \text{Hom}(B, U) \to \text{Hom}(A, U) \to \text{Hom}(C[-1], U) \to \cdots \]
2. Derived category of coherent sheaves and Fourier-Mukai transform

Let $X$ be a smooth projective variety over a field. We define $D^b(X) := D^b_{coh}(O_X - \text{mod}) \cong D^b_{coh}(Qcoh(X))$.

Let $f : X \rightarrow Y$ be a morphism of smooth projective varieties. Then $L_f^* : D^b(Y) \rightarrow D^b(X)$ and $Rf_* : D^b(X) \rightarrow D^b(Y)$ are well-defined adjoint functors. We also have an adjoint pair $\otimes^L$ and $R\text{Hom}$.

To simplify the notation we omit $R$ and $L$ from the notation from now on.

We will need the following two formulas:

1. **Projection formula** Let $f : X \rightarrow Y$ be a morphism of smooth projective varieties. Then we have a natural isomorphism in $D^b(Y)$:

   $f_* (\mathcal{F} \otimes f^*(\mathcal{G})) \cong f_*(\mathcal{F}) \otimes \mathcal{G}$.

2. **Base change formula** Given a cartesian diagram

   \[
   \begin{array}{ccc}
   X & \xrightarrow{a} & Y \\
   f \downarrow & & \downarrow g \\
   Z & \xrightarrow{b} & T
   \end{array}
   \]

   with $g$ (and thus also $f$) is flat. Then for any sheaf $\mathcal{F}$ on $Z$ we have a natural isomorphism in $D^b(Y)$

   $g^* b_*(\mathcal{F}) \cong a_* f^*(\mathcal{F})$.

   For any sheaf (so-called kernel) $\mathcal{F}$ on $X \times Y$ we may consider the Fourier-Mukai transform

   $\text{FM}_\mathcal{F} : D^b(Y) \rightarrow D^b(X)$

   given as the composition

   $\mathcal{E} \mapsto p_{1,*}(\mathcal{F} \otimes p_2^*(\mathcal{E}))$.

   More generally we could define a Fourier-Mukai transform $\text{FM}_{\mathcal{T},G} : D^b(Y) \rightarrow D^b(X)$ given any diagram

   \[
   \begin{array}{ccc}
   T & \xrightarrow{a} & X \\
   \downarrow b & & \downarrow \ \\
   Y & \xrightarrow{b} & T
   \end{array}
   \]

   and an object $\mathcal{G} \in D^b(T)$ by the formula:

   $\Phi(\mathcal{E}) := a_*(\mathcal{G} \otimes b^*\mathcal{E})$.

   However the next lemma shows that we get essentially the same class of functors.

**Lemma 2.1.** We have an isomorphism $\text{FM}_{\mathcal{T},G} \cong \text{FM}_\mathcal{F}$ where $\mathcal{F} = (a, b)_*(\mathcal{G})$.

We list some properties of the Fourier-Mukai transform.
Proposition 2.2. 0. Given a morphism $f : X \to Y$ consider its graph $\Gamma \subset X \times Y$. We have

\[ FM\Omega_\Gamma \simeq f^* \]
\[ FM\Omega'_\Gamma \simeq f_* . \]

1. $FM\mathcal{F} \circ FM\mathcal{G} \simeq FM_{\mathcal{F} \circ \mathcal{G}}$ where $\mathcal{F} \circ \mathcal{G} := p_{13,*}(p_{12}^*F \otimes p_{23}^*G)$

2. $FM\Omega_{\Delta} \simeq Id_{D^b(X)}$

3. $FM_{p_1^*(\mathcal{F}_1) \otimes p_2^*(\mathcal{F}_2)}(\mathcal{E}) \simeq \mathcal{F}_1 \otimes \Gamma(Y, \mathcal{F}_2 \otimes \mathcal{E})$

4. If $FM_{\mathcal{F}'}(\mathcal{E}) \to FM_{\mathcal{F}}(\mathcal{E}) \to FM_{\mathcal{F}''}(\mathcal{E}) \to FM_{\mathcal{F}'}(\mathcal{E})[1]$ is a distinguished triangle, then for any $\mathcal{E} \in D^b(Y)$

is a distinguished triangle.

Proof. To prove (0) let $i = (id, f) : X \to X \times Y$ so that $\mathcal{O}_\Gamma \simeq i_*\mathcal{O}_X$. Now we compute

\[ FM\mathcal{O}_\Gamma(\mathcal{E}) = p_{1,*}(i_*\mathcal{O}_X) \otimes p_{2}^*\mathcal{E} \simeq p_{1,*}(i_*i^*p_{2}^*\mathcal{E}) \simeq (p_2 \circ i)^*(\mathcal{E}) \simeq f^*(\mathcal{E}) \]

and

\[ FM\mathcal{O}'_\Gamma(\mathcal{E}) = p_{2,*}(i_*\mathcal{O}_X) \otimes p_{2}^*\mathcal{E} \simeq p_{2,*}(i_*i^*p_{2}^*\mathcal{E}) \simeq (p_2 \circ i)_*(\mathcal{E}) \simeq f_*(\mathcal{E}) . \]

To prove (1) we first show that

\[ FM\mathcal{F} \circ FM\mathcal{G} \simeq FM_{X \times Y \times Z,p_{12}^*(\mathcal{F}) \otimes p_{23}^*(\mathcal{G})} \]

(this uses the base change formula). Now from the Lemma above it follows that

\[ FM_{X \times Y \times Z,p_{12}^*(\mathcal{F}) \otimes p_{23}^*(\mathcal{G})}(\mathcal{E}) \simeq FM_{p_{13},(p_{12}^*(\mathcal{F}) \otimes p_{23}^*(\mathcal{G}))}(\mathcal{E}) \simeq FM_{\mathcal{F} \circ \mathcal{G}} . \]

(2) follows from (0) with $f = id$.

In (3) we use the isomorphism $FM_{\mathcal{O}_X}(\mathcal{E}) \simeq \mathcal{O}_X \otimes \Gamma(Y, \mathcal{E})$ which follows from the base change.

Finally (4) is obvious because all the functors involved are triangulated. \[ \square \]

3. SEMIORTHOGONAL DECOMPOSITIONS

Let $\mathcal{C}$ be a triangulated category, and let $\mathcal{A}$ be its full subcategory.

Definition 3.1. The right orthogonal to $\mathcal{A}$ is defined as:

$\mathcal{A}^\perp = \{ \mathcal{C} \in \mathcal{C} : Hom(A, C) = 0 \ \forall A \in \mathcal{A} \}$.

Similarly the left orthogonal to $\mathcal{A}$ is defined as:

$\perp \mathcal{A} = \{ \mathcal{C} \in \mathcal{B} : Hom(C, A) = 0 \ \forall A \in \mathcal{A} \}$.

Example 3.2. Let $\mathcal{C} = D^b(X)$, $\mathcal{A} = \langle \mathcal{O}_X \rangle$. Then the right orthogonal to $\mathcal{A}$ consists of $\Gamma_X$-acyclic complexes.

Lemma 3.3. Let $\mathcal{A}, \mathcal{B}$ be triangulated subcategories of $\mathcal{C}$ with the inclusion functors denoted by $i$ and $j$ respectively. Assume that $Hom(B, A) = 0$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. Then the following conditions are equivalent:
1. \( A \) and \( B \) generate \( C \) as a triangulated category
2. For each \( X \in C \) there exists a distinguished triangle
   \[
   B \to X \to A \to B[1]
   \]
   with \( A \in A \), \( B \in B \).
3. \( B = ^\perp_A A \) and there exists a functor \( i^* : C \to A \) which is left adjoint to \( i : A \to C \)
4. \( A = B^\perp \) and there exists a functor \( j^* : C \to B \) which is right adjoint to \( j : B \to C \)

If these conditions are satisfied \( A \) is called left admissible, \( B \) is called right admissible and we say that we have a semiorthogonal decomposition \( A = \langle A, B \rangle \). In this case the components \( A, B \) of \( X \) in 2. are defined uniquely up to an isomorphism.

Proof. It is obvious that (2.) implies (1.). The reverse implications is proved as follows. Consider the subcategory \( C' \subset C \) consisting of objects \( X \) that admit a decomposition into distinguished triangle as in (2.). \( C' \) contains both \( A \) and \( B \) and thus it is sufficient to prove that \( C' \) is triangulated. \( C' \) is obviously closed under shifts. The fact that \( C' \) is closed under cones is the generalized octahedron axiom giving a diagram of distinguished triangles:

\[
\begin{array}{ccc}
B & \to & X & \to & A \\
\downarrow f_a & & \downarrow f & & \downarrow f_A \\
B' & \to & X' & \to & A'
\end{array}
\]

\[
\begin{array}{ccc}
\text{cone}(f_B) & \to & \text{cone}(f) & \to & \text{cone}(f_A)
\end{array}
\]

It is easy to see that (4.) implies (2.). Indeed the required triangle is

\[
j^! X \to X \to \text{cone}(j^! X \to X)
\]

with \( \text{Hom}(B, X) \cong \text{Hom}(B, j^! X) \) implying \( A = \text{cone}(j^! X \to X) \in A = B^\perp \).

Let us prove that (2.) implies (4.) We first note that in any diagram

\[
\begin{array}{ccc}
B & \to & X & \to & C \\
\downarrow & & \downarrow & & \downarrow \\
B' & \to & X' & \to & C'
\end{array}
\]

the dotted arrows uniquely exist. Therefore the assignment \( X \mapsto i^! X := B \) is functorial once we fix a \( B \) for each \( X \).

Similary, (3.) is equivalent to (2.)

Example 3.4. Let \( C = D^b(X) \) and \( A \) be the triangulated category generated by skyscraper sheaves \( O_x, x \in X \).

Then \( A \) is neither right nor left admissible. Indeed both orthogonals to \( A \) are zero, whereas \( A \neq C \).

Example 3.5. We will use the Serre functor in the next lecture to show that a variety \( X \) with \( K_X = 0 \) has no nontrivial admissible subcategories in \( D^b(X) \).
Remark 3.6. Can we have fully orthogonal decompositions of $D^b(X)$? No, unless $X$ is disconnected. Indeed if $D^b(X) = A \times B$, then we can write $O_X$ as a direct sum of two sheaves which implies that $X$ is reducible.

Definition 3.7. If $A_1, \ldots, A_r$ are triangulated subcategories of $C$ $Hom(A_j, A_i) = 0$ for $j > i$ and $A_i$ generate $C$, then we say that $C$ admits a semi-orthogonal decomposition $C = \langle A_1, \ldots, A_r \rangle$.

Saturatedness?

4. Exceptional collections

Definition 4.1. An object $E \in C$ is called exceptional if $Hom^*(E, E) = C[0]$.

Example 4.2. The structure sheaf $O_X \in D^b(X)$ is exceptional if and only if $h^i(X) = 0$ for $i > 0$. In this case any line bundle $L$ on $X$ is exceptional.

Definition 4.3. A sequence $E_1, \ldots, E_r$ of exceptional objects is called an exceptional sequence if $Hom^*(E_j, E_i) = 0$ for $j > i$.

An exceptional collection is called full if it generates $C$, or equivalently, $\langle \langle E_1 \rangle, \ldots, \langle E_r \rangle \rangle$ is a semi-orthogonal decomposition of $C$.

Proposition 4.4. Let $E_1, \ldots, E_r$ be an exceptional collection of sheaves on $X$. Assume that there exists a resolution of the diagonal on $X \times X$ of the form

$$0 \to p_1^* E_1 \otimes p_2^* F_1 \to \cdots \to p_1^* E_r \otimes p_2^* F_r \to O_\Delta \to 0$$

where $F_i, i = 1, \ldots, r$ is also a sequence of sheaves on $X$ (not necessarily exceptional). Then the collection $E_i$ is full.

Proof. The resolution above is equivalent to a list of short exact sequences:

$$0 \to H_{k-1} \to p_1^* E_k \otimes p_2^* F_k \to H_k \to 0, \quad k = 1, \ldots, r$$

$H_0 = 0, H_r = O_\Delta$

We prove the following statement by induction on $k$:

$$FM_{H_k}(C) \subset \langle E_1, \ldots, E_k \rangle.$$ 

Indeed, the statement is trivial for $k = 0$ and the distinguished triangle

$$FM_{H_{k-1}}(\mathcal{E}) \to E_k \otimes \Gamma(F_k \otimes \mathcal{E}) \to FM_{H_k}(\mathcal{E}) \to FM_{H_{k-1}}(\mathcal{E})[1]$$

is used to make the induction step. Substituting $k = r$ gives the result. □

Theorem 4.5. The sequences $O(-n), \ldots, O(-1), O$ and $\Omega^n(n), \ldots, \Omega^1(1), O$ are full exceptional collections on $P^n$.

It is well-known that the appropriate cohomology groups vanish so that the first sequence is exceptional. The second sequence can be seen to be exceptional by induction using the exact sequence:

$$0 \to \Omega^1 \to V^*(1) \to O \to 0$$

and its exterior powers

$$0 \to \Omega^k \to \wedge^k V^*(k) \to \Omega^{k-1} \to 0.$$
Lemma 4.6. We have the following resolution:

\[ 0 \to p_1^*(\Omega^2(n)) \otimes p_2^*(\mathcal{O}(-n)) \to \cdots \to p_1^*(\Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1)) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_{\Delta} \to 0 \]

**Proof.** It suffices to find a regular section \( s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*(T(-1)) \otimes p_2^*(\mathcal{O}(1))) \) which vanishes precisely on the diagonal \( \Delta \subset \mathbb{P}^n \times \mathbb{P}^n \). Indeed, in this case the Koszul complex gives a resolution of the diagonal as required. Indeed if \( \mathcal{E} = p_1^*(\Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1)) \), we may consider the following complex:

\[ 0 \to \wedge^2 \mathcal{E} \to \cdots \to \wedge^n \mathcal{E} \to \mathcal{E} \to \mathcal{O}_{\Delta} \to 0 \]

where each differential is a contraction with \( s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{E}^\vee) \). The standard theorem of homological algebra says that the complex above is exact if the section \( s \) is regular (that is, generic in a certain sense).

To find such a regular section \( s \) we start with the Euler exact sequence on \( \mathbb{P}^n \):

\[ 0 \to \mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to T(-1) \to 0 \]

and then consider the following composition:

\[ \phi : p_2^*\mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to p_1^*T(-1). \]

At the point \( (l_1, l_2) \in \mathbb{P}^n \times \mathbb{P}^n \) we have

\[ \phi(l_1, l_2) : l_1 \subset V \to V/l_1. \]

In particular, \( \phi(l_1, l_2) = 0 \) if and only if \( l_1 = l_2 \).

Let \( s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*(T(-1)) \otimes p_2^*(\mathcal{O}(1))) \) be the section corresponding to \( \phi \). Then \( s \) vanishes precisely along the diagonal and \( s \) is regular since of the codimension of the diagonal is equal to \( n \) which is the rank of the bundle. \( \Box \)