DERIVED CATEGORIES: LECTURE 2

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References

[Bo] Alexey Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat., 53:1 (1989),

[BK] Alexey Bondal, Mikhail Kapranov, *Representable functors, Serre functors, and mutations*, Izv. Akad. Nauk SSSR Ser. Mat., 53:6 (1989),

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Proposition 0.1. *Let $E_1, \dots E_r$ be an exceptional collection of sheaves on X. Assume that there exists a resolution of the diagonal on $X \times X$ of the form*

$$0 \to p_1^* E_1 \otimes p_2^* F_1 \to \cdots \to p_1^* E_r \otimes p_2^* F_r \to \mathcal{O}_\Delta \to 0$$

where F_i, $i = 1, \dots, r$ is also a sequence of sheaves on X (not necessarily exceptional). Then the collection E_i is full.

Lemma 0.2. *We have the following resolution:*

$$0 \to \Omega^n(n) \boxtimes \mathcal{O}(-n) \to \cdots \to \Omega^1(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

Proof. It suffices to find a regular section $s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*(T(-1)) \otimes p_2^*(\mathcal{O}(1)))$ which vanishes precisely on the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. Indeed, in this case the Koszul complex gives a resolution of the diagonal as required. Indeed if $\mathcal{E} = p_1^*(\Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1))$, we may consider the following complex:

$$0 \to \wedge^n \mathcal{E} \to \cdots \wedge^2 \mathcal{E} \to \mathcal{E} \to \mathcal{O}_\Delta \to 0$$

where each differential is a contraction with $s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{E}^\vee)$. The standard theorem of homological algebra says that the complex above is exact if the section s is regular (that is, generic in a certain sense).

To find such a regular section s we start with the Euler exact sequence on $\mathbb{P}^n$:

$$0 \to \mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to T(-1) \to 0$$

and then consider the following composition:

$$\phi : p_2^* \mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to p_1^* T(-1).$$

At the point $(l_1, l_2) \in \mathbb{P}^n \times \mathbb{P}^n$ we have

$$\phi_{(l_1, l_2)} : l_1 \subset V \to V/l_1.$$

In particular, $\phi_{(l_1,l_2)} = 0$ if and only if $l_1 = l_2$.

Let $s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*(T(-1)) \otimes p_2^* \mathcal{O}(1))$ be the section corresponding to ϕ. Then s vanishes precisely along the diagonal and s is regular since of the codimension of the diagonal is equal to n which is the rank of the bundle. □

Triangulated categories in this lecture are assumed to be k-linear over some field k and of finite type, that is

$$\sum_{i\in\mathbb{Z}} dim\, Hom(A, B[i]) < +\infty$$

for all objects A, B. Derived categories of coherent sheaves $D^b(X)$ for smooth projective X are examples of such categories.

We call a subcategory of a triangulated category admissible if it is both left and right admissible.

1. Subcategories generated by exceptional collections are admissible

Lemma 1.1. *Let $E \in D^b(X)$ be an exceptional object. Then $\mathcal{E} = \langle E\rangle \subset D^b(X)$ is admissible. The adjoint functors to $i : \mathcal{E} \to D^b(X)$ are given as*

$$i^!(F) := E \otimes RHom(E, F)$$
$$i^*(F) := E \otimes RHom(F, E)^*$$

Proof. Let $E \otimes K^\bullet \in \mathcal{E}$, $K^\bullet \in D^b(Vect/k)$. We have

$$\begin{aligned} Hom_{\mathcal{E}}(E \otimes K^\bullet, i^! F) &= Hom_{\mathcal{E}}(E \otimes K^\bullet, E \otimes RHom(E, F)) \simeq \\ &\simeq Hom_{D^b(Vect/k)}(K^\bullet, RHom(E, F)) \simeq \\ &\simeq Hom_{D^b(Vect/k)}(K^\bullet, R\Gamma(\underline{RHom}(E, F))) \simeq \\ &\simeq Hom_{D^b(X)}(K^\bullet \otimes \mathcal{O}_X, \underline{RHom}(E, F)) \simeq \\ &\simeq Hom_{D^b(X)}(K^\bullet \otimes E, F). \end{aligned}$$

Similarly

$$\begin{aligned} Hom_{\mathcal{E}}(i^*(F), E \otimes K^\bullet) &= Hom_{\mathcal{E}}(E \otimes RHom(F, E)^*, E \otimes K^\bullet) \simeq \\ &\simeq Hom_{D^b(Vect/k)}(RHom(F, E)^*, K^\bullet) \simeq \\ &\simeq H^0(RHom(RHom(F, E)^*, K^\bullet)) \simeq \\ &\simeq H^0(RHom(F, E) \otimes K^\bullet) \simeq \\ &\simeq H^0(RHom(F, E \otimes K^\bullet)) \simeq \\ &\simeq Hom_{D^b(X)}(F, E \otimes K^\bullet). \end{aligned}$$

□

If $E \in D^b(X)$ be an exceptional object. Then for any $F \in D^b(X)$, we can consider projections of F onto the left and right ortohogonals to E.

That is, we consider the following two triangles:

$$E \otimes RHom(E, F) \to F \to L_E(F)$$
$$R_E(F) \to F \to E \otimes RHom(F, E)^*.$$

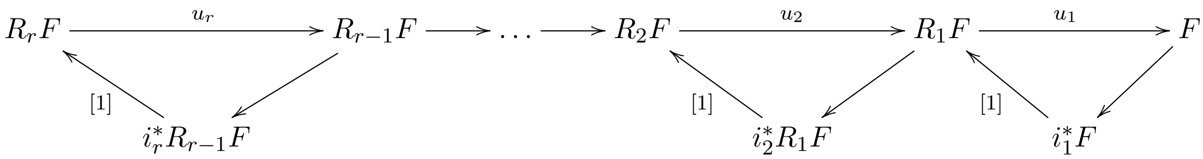
Definition 1.2. $L_E(F) \simeq cone(E \otimes RHom(E, F) \to F)$ is called the left mutation of F with respect to E.

$R_E(F) \simeq cone(F \to E \otimes RHom(F, E)^*)[-1]$ is called the right mutation of F with respect to E.

Note that $L_E(F)$ lies in the right orthogonal to E whereas $R_E(F)$ lies in the left orthogonal to E.

Proposition 1.3. *Let $\mathcal{E} = \langle E_1, \dots, E_r \rangle \subset D^b(X)$ be generated by an exceptional collection. Then $\mathcal{E}$ is admissible.*

Proof. Let $F \in D^b(X)$. Let $R_j F = R_{E_j} R_{E_{j-1}} \dots R_{E_1} F$ be the composition of j right mutations which together give rise to the following filtration of F:



(here i_j denotes the embedding $\langle E_j \rangle \subset D^b(X)$ and i_j^* is the corresponding left adjoint functor).

We prove by induction that $R_j F \in {}^\perp\langle E_1, \dots, E_j \rangle$. Indeed, for $j = 1$ we get $R_1 F = R_{E_1} F \in {}^\perp\langle E_1 \rangle$ by definition. Now since $R_j F = cone(R_{j-1}F \to i_j^* R_{j-1} F)[-1]$ and both terms $R_{j-1}F$ and $i_j^* R_{j-1} F$ lie in ${}^\perp\langle E_1, \dots, E_{j-1} \rangle$ it follows that

$$R_j F \in {}^\perp\langle E_1, \dots, E_{j-1} \rangle \cap {}^\perp\langle E_j \rangle = {}^\perp\langle E_1, \dots, E_j \rangle .$$

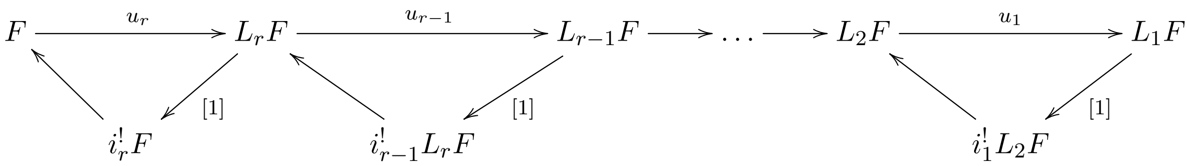
In particular $R_r F \in {}^\perp\mathcal{E}$.

Consider $u = u_r \circ \dots \circ u_2 \circ u_1 : R_r F \to F$. It follows from Lemma below that the term $F' = cone(u)$ in the triangle

$$R_r F \to F \to F' \to R_r F[1]$$

lies in $\mathcal{E}$. Thus $\mathcal{E}$ is left admissible.

Similarly we can use left mutations:

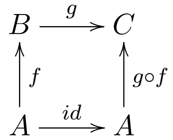


Then $cone(u_1 \circ u_2 \circ \dots \circ u_r : F \to L_1 F) \in \mathcal{E}$, $L_1 F \in \mathcal{E}^\perp$ and $\mathcal{E}$ is right admissible. □

Lemma 1.4. *Let $f : A \to B$, $g : B \to C$ be morphisms in a triangulated category. Then $cone(g \circ f)$ fits into a distinguished triangle*

$$cone(f) \to cone(g \circ f) \to cone(g) \to cone(f)[1]$$

Proof. We start with a commutative square



The octahedron axiom implies that this square can be included into a diagram with distinguished rows and columns

$$\begin{array}{ccccc} cone(f) & \longrightarrow & cone(g\circ f) & \longrightarrow & cone(g)\\ \uparrow & & \uparrow & & \uparrow \\ B & \xrightarrow{g} & C & \longrightarrow & cone(g) \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g\circ f} & & \uparrow \\ A & \xrightarrow{id} & A & \longrightarrow & 0 \end{array}$$

□

2. Mutations of exceptional collections

Let $\mathcal{A}$ be an admissible subcategory in $\mathcal{C}$. In the most general form mutations are the corresponding projections $L_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{\perp}$ and $R_{\mathcal{A}} : \mathcal{A} \to {}^{\perp}\mathcal{A}$.

Lemma 2.1. *$L_{\mathcal{A}}\big|_{{}^{\perp}\mathcal{A}}$ and $R_{\mathcal{A}}\big|_{\mathcal{A}^{\perp}}$ are mutually inverse equivalences between the two orthogonals to $\mathcal{A}$.*

Proof. We check that $L_{\mathcal{A}}R_{\mathcal{A}}\big|_{\mathcal{A}^{\perp}}$ is isomorphic to identity, the proof for other composition goes exactly the same. Let $B \in \mathcal{A}^{\perp}$. Then $B' = R_{\mathcal{A}}(B)$ is defined from the distinguished triangle

$$B' \to B \to A_1 \to B'[1]. \tag{2.1}$$

To compute $B'' = L_{\mathcal{A}}(B')$ we need a distinguished triangle

$$A_2 \to B' \to B'' \to A_2[1].$$

However from the triangle 2.1 we obtain:

$$A_1[-1] \to B' \to B \to A_1.$$

Thus we get $A_2 \simeq A_1[-1]$, $B'' \simeq B$. □

Let $E_\bullet = (E_1, \ldots, E_r)$ be an exceptional collection in $\mathcal{C}$. For each $i = 1, \ldots, r-1$ we define left and right i-th mutation of $E_\bullet$ as follows:

$$L_i(E_\bullet) = (E_1, \ldots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_n)$$
$$R_i(E_\bullet) = (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \ldots, E_n)$$

Proposition 2.2. *1. $L_i(E_\bullet)$ and $R_i(E_\bullet)$ are exceptional collections. If $E_\bullet$ is full, then so are $L_i(E_\bullet)$ and $R_i(E_\bullet)$.*

2. Consider the action of the group generated by all L_i and R_j on the set of isomorphism classes of exceptional collections. Then the following relations are satisfied:

$$L_iR_i = R_iL_i = id$$
$$L_{i+1}L_iL_{i+1} = L_iL_{i+1}L_i$$
$$R_{i+1}R_iR_{i+1} = R_iR_{i+1}R_i$$
$$L_iL_j = L_jL_i,\ R_iR_j = R_jR_i,\ \ |i-j| \geq 2.$$

This gives a braid group action on the set of isomorphism classes of exceptional collections.

Proof. The first claim follows easily from definitions and Lemma 2.1.

The first formula in (2) also follows from Lemma 2.1. The fourth formula in (2) is obvious. Let us prove the second formula. We assume for simplicity that $E_\bullet = (E_1, E_2, E_3)$ and $i = 1$. In this case

$$L_1L_2L_1E_\bullet = (L_{E_1}L_{E_2}E_3, L_{E_1}E_2, E_1)$$
$$L_2L_1L_2E_\bullet = (L_{L_{E_1}E_2}L_{E_1}E_3, L_{E_1}E_2, E_1).$$

On the other hand

$$L_{E_1}L_{E_2}E_3 \simeq L_{\langle E_1,E_2\rangle}E_3 \simeq L_{\langle L_{E_1}E_2,E_1\rangle}E_3 \simeq L_{L_{E_1}E_2}L_{E_1}E_3.$$

□

Example 2.3. Let $X = \mathbb{P}^n$. We prove that

$$L_{\langle \mathcal{O},\dots,\mathcal{O}(k)\rangle}\mathcal{O}(k+1) \simeq \Omega^{k+1}(k+1)[k+1]. \tag{2.2}$$

In particular "reverting" the collection $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ we obtain a collection

$$\Omega^n(n)[n], \dots, \Omega^1(1)[1], \mathcal{O}.$$

Our computations rely on Euler's short exact sequence:

$$0 \to \Omega^k(j) \to \Lambda^k V^* \otimes \mathcal{O}(j-k) \to \Omega^{k-1}(j) \to 0. \tag{2.3}$$

Consider the associated long exact sequence of cohomology groups:

$$0 \to H^0(\Omega^k(j)) \to H^0(\Lambda^k V^* \otimes \mathcal{O}(j-k)) \to H^0(\Omega^{k-1}(j)) \to$$
$$\dots$$
$$H^p(\Omega^k(j)) \to H^p(\Lambda^k V^* \otimes \mathcal{O}(j-k)) \to H^p(\Omega^{k-1}(j)) \to \dots$$

For $0 \le j < k \le n$ all the middle terms vanish. Then descending induction on k shows that the sheaf $\Omega^k(j)$ is acyclic for $1 \le j \le k$ (the base case is $k = n$, where $\Omega^n(j) = \mathcal{O}(-n-1+j)$ is indeed acyclic in the give range). It follows then from the same long exact sequence with $k = j$ that $H^*(\Omega^{k-1}(k)) = \Lambda^k V^*[0]$.

Setting $k = j$ in the sequence (2.3) we get a distinguished triangle in $D^b(\mathbb{P}^n)$:

$$\Omega^k(k) \to \Lambda^k V^* \otimes \mathcal{O} \to \Omega^{k-1}(k) \to \Omega^k(k)[1].$$

Using this triangle and the definition of left mutation we compute

$$L_{\mathcal{O}}\Omega^{k-1}(k) = cone(R\Gamma(\Omega^{k-1}(k)) \otimes \mathcal{O} \to \Omega^{k-1}(k)) \simeq \Omega^k(k)[1]. \tag{2.4}$$

Finally (2.2) is proven by induction using (2.4):

$$\begin{aligned} L_{\langle \mathcal{O},\dots,\mathcal{O}(k)\rangle}\mathcal{O}(k+1) &\simeq L_{\langle \mathcal{O}\rangle}L_{\langle \mathcal{O}(1),\dots,\mathcal{O}(k)\rangle}\mathcal{O}(k+1) \simeq \\ &\simeq L_{\langle \mathcal{O}\rangle}(L_{\langle \mathcal{O},\dots,\mathcal{O}(k-1)\rangle}\mathcal{O}(k))(1) \simeq \\ &\simeq L_{\langle \mathcal{O}\rangle}\Omega^k(k+1)[k] \simeq \\ &\simeq \Omega^{k+1}(k+1)[k+1]. \end{aligned}$$

3. The Serre functor

Definition 3.1. Let $\mathcal{C}$ be an Ext-finite triangulated category. We say that a covariant auto-equivalence $S : \mathcal{C} \to \mathcal{C}$ is a Serre functor if we have a bifunctorial isomorphism

$$Hom(A, B)^* \simeq Hom(B, SA), \; A, B \in \mathcal{C}$$

Proposition 3.2. *If X is a smooth projective n-dimensional variety, then $D^b(X)$ admits a Serre functor given as*

$$S(\mathcal{F}^\bullet) = \mathcal{F}^\bullet \otimes \omega_X[n]$$

($\omega_X = \wedge^n \Omega^1_X$ is the determinant of the cotangent sheaf).

Proof. Recall that if $\mathcal{F}$ is a coherent sheaf on X, the Serre duality theorem [Har],III.7 tells that there is a natural isomorphism

$$H^i(X, \mathcal{F}) \simeq Ext^{n-i}(\mathcal{F}, \omega_X)^*.$$

In fact these isomorphisms lift to a natural quasiisomorphism of complexes

$$R\Gamma(X, \mathcal{F}) \simeq RHom(\mathcal{F}, \omega_X[n])^*.$$

Note that dualizing the complex puts a component of degree i into degree $-i$.

Now let $\mathcal{F}_1, \mathcal{F}_2 \in D^b(X)$ and let

$$\mathcal{F} = \underline{RHom}(\mathcal{F}_1, \mathcal{F}_2) \cong \mathcal{F}_2 \otimes \mathcal{F}_1^\vee,$$

where $\mathcal{F}_1^\vee = \underline{RHom}(\mathcal{F}_1, \mathcal{O}_X)$. We have

$$RHom(\mathcal{F}_1, \mathcal{F}_2) \cong RHom(\mathcal{O}, \mathcal{F}) \cong RHom(\mathcal{F}, \omega_X[n])^* \cong RHom(\mathcal{F}_2, \mathcal{F}_1 \otimes \omega_X[n])^*.$$

□

Proposition 3.3. *1. $\mathcal{C}$ admits a Serre functor if and only if all contravariant functors $Hom(A, \bullet)^*, A \in \mathcal{C}$ and all covariant functors $Hom(\bullet, A)^*, A \in \mathcal{C}$ are representable.*

2. If $\mathcal{C}$ admits a Serre functor S, then S is a triangulated functor and S is unique up to a canonical isomorphism.

3. If $\mathcal{A}$ is an admissible subcategory in $\mathcal{C}$ and $\mathcal{C}$ admits a Serre functor, then $\mathcal{A}$ admits a Serre functor and it is given by

$$S_{\mathcal{A}} = i^! S_{\mathcal{C}}.$$
$$S_{\mathcal{A}}^{-1} = i^* S_{\mathcal{C}}^{-1}.$$

4. Let $\mathcal{A} \subset \mathcal{C}$ be a triangulated subcategory and assume that S is a Serre functor on $\mathcal{C}$. Then

$$S({}^\perp\mathcal{A}) = \mathcal{A}^\perp$$
$$S^{-1}(\mathcal{A}^\perp) = {}^\perp\mathcal{A}.$$

5. If $E_\bullet = (E_1, E_2, \ldots, E_n)$ is a full exceptional collection, then

$$L_1 L_2 \ldots L_{n-1} E_\bullet = (S(E_n), E_1, \ldots, E_{n-1})$$
$$R_{n-1} R_{n-2} \ldots R_1 E_\bullet = (E_2, \ldots, E_n, S^{-1}(E_1))$$

Proof. For (1) and (2) see [BK]. (3) and (4) follow easily from definitions. (5) follows from (4) applied to the subcategory generated by $n-1$ objects. □

Definition 3.4. A category $\mathcal{C}$ is called an n-Calabi-Yau category if the functor $[n]$ is a Serre functor.

Lemma 3.5. *1. Let $\mathcal{C}$ be a Calabi-Yau category. Then any semi-orthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is fully orthogonal, that is $\mathcal{C} \cong \mathcal{A} \oplus \mathcal{B}$.*

2. If X is a Calabi-Yau variety, then $D^b(X)$ admits no non-trivial (left or right) admissible subcategories.

Proof. By definition of what an admissible subcategory is, it suffices to show that any semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is trivial.

Let $A \in \mathcal{A}, \mathcal{B} \in B$. Then

$$Hom(A, B) \cong Hom(B, S(A))^* = Hom(B, A[n]) = 0.$$

This shows the first claim.

To prove the second claim, note that since X is connected $\mathcal{O}_X$ admits no notrivial direct summands, thus if $D^b(X) \cong \mathcal{A} \oplus \mathcal{B}$, then $\mathcal{O}_X$ must lie in either $\mathcal{A}$ or $\mathcal{B}$. Assume that $\mathcal{O}_X \in \mathcal{A}$. The same reasoning applies to each $\mathcal{O}_x, x \in X$ as well. However, since $Hom(\mathcal{O}_X, \mathcal{O}_x) \neq 0$ for all $x \in X$, all $\mathcal{O}_x \in \mathcal{A}$. Now $\mathcal{B}$ is orthogonal to all skyscraper sheaves and their shifts, thus $\mathcal{B} = 0$. □

Proposition 3.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between triangulated categories. Assume that both $\mathcal{C}$ and $\mathcal{D}$ have Serre functors.*

1. If F admits a left adjoint $G : \mathcal{D} \to \mathcal{C}$, then it also admits a right adjoint $H : \mathcal{D} \to \mathcal{C}$, given as

$$H = S_C \circ G \circ S_D^{-1}.$$

2. If F admits a right adjoint $H : \mathcal{D} \to \mathcal{C}$, then it also admits a left adjoint $G : \mathcal{D} \to \mathcal{C}$, given as

$$G = S_C^{-1} \circ H \circ S_D.$$

Proof. The proof is straightforward using the definitions. □

Example 3.7. Let $f : X \to Y$ be a morphism of smooth projectvie varieties. Then $f_* : D^b(X) \to D^b(Y)$ admits a left adjoint f^*. By the Proposition above f_* also admits the right adjoint functor, which is denoted $f^!$. We have

$$\begin{aligned} f^!(\mathcal{G}^\bullet) &:= S_X \circ f^* \circ S_Y^{-1}(\mathcal{G}^\bullet) \simeq \\ &\simeq S_X \circ f^*(\mathcal{G} \otimes \omega_Y^\vee[-dimY]) \simeq \\ &\simeq S_X(f^*(\mathcal{G}) \otimes f^*(\omega_Y^\vee)[-dimY]) \simeq \\ &\simeq f^*(\mathcal{G}) \otimes f^*(\omega_Y^\vee)[-dimY] \otimes \omega_X[dimX] \simeq \\ &\simeq f^*(\mathcal{G}) \otimes \omega_{X/Y}[dimX - dimY], \end{aligned}$$

where $\omega_{X/Y} = \omega_X \otimes f^*(\omega_Y^\vee)$.

In particular, if $f = i : X \to Y$ is a closed embedding of codimension r, then

$$i^!(\mathcal{G}^\bullet) = i^*(\mathcal{G}^\bullet) \otimes \wedge^r \mathcal{N}_{X/Y}^\vee[-r].$$

On the other hand if $f = p : X \to Spec(k)$ is the projection to a point, then

$$D_X := p^!(\mathcal{O}_{Spec(k)}) \simeq \omega_X[dim(X)]$$

is the so-called dualizing complex.

Corollary 3.8. *If $\mathcal{A}$ is a left (or right) admissible subcategory in $\mathcal{C}$ and both $\mathcal{A}$ and $\mathcal{C}$ have Serre functors, then $\mathcal{A}$ is admissible.*

Remark 3.9. In the situation of the Corollary above, we do not usually have $S_{\mathcal{A}} = S_{\mathcal{C}}\big|_{\mathcal{A}}$. Indeed, when this is the case then one can prove that $\mathcal{A}^{\perp} = {}^{\perp}\mathcal{A}$, and thus $\mathcal{C} \simeq \mathcal{A} \oplus \mathcal{A}^{\perp}$ is completely orthogonal.