DERIVED CATEGORIES: LECTURE 4

EVGENY SHINDER

References

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[BO2] Alexey Bondal, Dmitri Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. 125 (2001), no. 3, 327–344.

1. Overview

One of the approaches to non-commutative geometry is to consider the category of sheaves $Shv(X)$ on a given geometric object X as a primary object of study.

That is, by definition a generalized space would be a category $\mathcal{C}$ with some properties. We think of $\mathcal{C}$ as the category of sheaves on a space. It is then natural to ask which properties of the original geometric object X can be extracted from the category $Shv(X)$, and ultimately, whether X itself can be reconstructed.

Precise answer to this question depends on the model we choose. Here are some examples.

1. *Rosenberg's spectrum of an abelian category.* If $\mathcal{A}$ is an abelian category, one can attach to it a locally ringed space $Spec(\mathcal{A})$ in such a way that if $\mathcal{A}_X = QCoh(X)$ on a scheme X, then $Spec(\mathcal{A}_X) \cong X$. Thus by passing to the abelian category of sheaves we do not lose information.

2. *Balmer's tensor triangulated geometry.* If $\mathcal{K}$ is a tensor triangulated category, then one can associate to it a locally ringed space $Spec(\mathcal{K})$ which also recovers the scheme if $\mathcal{K}$ is the category of sheaves. Roughly speaking, points of $Spec(\mathcal{K})$ are $\otimes$-ideals in $\mathcal{K}$.

In particular in both cases above if the corresponding categories of sheaves on X and Y are equivalent, then X and Y are isomorphic.

This is not the case for $D^b(X)$ (considered as usual as a triangulated category). Indeed there are examples of $K3$ surfaces and abelian varieties for which derived equivalence does not imply isomorphism.

On the other hand if $K_X > 0$ or $K_X < 0$, then Bondal and Orlov proved that it is possible to reconstruct X from $D^b(X)$.

2. Abelian varieties and $K3$ surfaces

Sometimes a moduli space Y of sheaves or bundles with prescribed properties on a variety X turns out to be derived equivalent to X without being isomorphic to X. In this section we indicate how this works for Abelian varieties (Mukai) and $K3$ surfaces (Mukai, Orlov, Bridgeland).

Theorem 2.1. [Br] *Let $\mathcal{P}$ be a vector bundle on $X \times Y$. Assume that for all $y \in Y$ the stalks $\mathcal{P}_y \in D^b(X)$ satisfy $Hom^0(\mathcal{P}_y, \mathcal{P}_y) = \mathbb{C}$, and are pairwise orthogonal: $Ext^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$ if $y_1 \neq y_2$. Then the Fourier-Mukai functor $FM_{\mathcal{P}} : D^b(Y) \to D^b(X)$ is an equivalence. If in addition $\mathcal{P}_y = \mathcal{P}_y \otimes \omega_X$ for all $y \in Y$, then $FM_{\mathcal{P}}$ is a derived equivalence*[1]*.*

Note that the conditions of the Theorem are necessary in order for $FM_{\mathcal{P}}$ to induce a derived equivalence $D^b(Y) \to D^b(X)$. Indeed by definition we have

$$FM_{\mathcal{P}}(\mathcal{O}_y) = \mathcal{P}_y,$$

and then since $Hom(k(y), k(y)) = k$ and $Hom(k(y_1), k(y_2))$, $y_1 \neq y_2$, the same must hold true for $\mathcal{P}_y$'s. Also, since ω_X is up to a shift the Serre functor, and the Serre functor is unique, $k(y) \otimes \omega_Y = k(y)$ implies $\mathcal{P}_y = \mathcal{P}_y \otimes \omega_X$.

Let A be a complex abelian variety. By definition $\widehat{A} = Pic^0(A)$ is the fine moduli space of A-invariant line bundles on A:

$$Pic^0(A)(k) = \{L \in Pic(A) : t_x^*(L) \cong L \ \forall x \in A\}$$

We take $\mathcal{P}$ to be the normalized Poincare bundle on $A \times \widehat{A}$.

Corollary 2.2. *A and $\widehat{A}$ are derived equivalent.*

Proof. For any $y \in \widehat{A}$ we have $Hom_A(\mathcal{P}_y, \mathcal{P}_y) \cong Hom_A(\mathcal{O}_A, \mathcal{O}_A) = \mathbb{C}$. On the other hand if

$$Hom^*_A(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = H^*(A, \mathcal{P}_{y_2} \otimes \mathcal{P}_{y_1}^\vee)$$

is not zero, then one can show that $\mathcal{P}_{y_2} \otimes \mathcal{P}_{y_1}^\vee$ is a trivial bundle, so that $\mathcal{P}_{y_1} \cong \mathcal{P}_{y_2}$, thus $y_1 = y_2$. □

Corollary 2.3. *Let X be a $K3$ surface and assume that the there exists a fine compact two dimensional moduli space Y of stable vector bundles on X. Then Y is derived equivalent to X and in fact Y is also a $K3$ surface.*

3. $K_X > 0$ or $K_X < 0$

Theorem 3.1. *Let X and X' be smooth projective varieties and assume that $K_X > 0$ (K_X is ample) or $K_X < 0$ ($-K_X$ is ample). Suppose that derived categories $D^b(X)$ and $D^b(X')$ are equivalent. Then X and X' are isomorphic.*

Remark 3.2. In the proof of Theorem 3.1 one only uses the graded structure of triangulated categories, that is the shift functor.

[1] And this condition is automatically satisfied when $\omega_X = \mathcal{O}_X$, that is X is a Calabi-Yau variety.

The hardest step in the proof of Theorem 3.1 is to show that $K_{X'}$ (resp. $-K_{X'}$) is also ample. If we assumed that, the proof would be very easy (see Step 5 in the proof of Theorem 3.1 below). We will need the following characterization of ampleness.

Proposition 3.3. *Let X be a projective variety and L a line bundle on X. The following conditions are equivalent:*

1. *L is ample*
2. *The canonical morphism $X \to Proj\Big(\oplus_{j\geq 0}\Gamma(X, L^{\otimes j})\Big)$ is an isomorphism*
3. *The system of open sets $\{x \in X | s_x \neq 0\}$ for $s \in \Gamma(X, L^{\otimes j})$, $j \in \mathbb{Z}$ forms a basis of Zarisky topology on X, that is for any closed $Z \subset X$ and $x \notin Z$ there exists a section $s \in \Gamma(X, L^{\otimes j})$ such that s vanishes on Z and does not vanish at x.*

Proof. (1) $\Longrightarrow$ (2) By Hartshorne Exercise 5.13, the graded ring under the $Proj$ does not change when we replace L by some power of L. Therefore we may assume that L is very ample, that is there exists a closed embedding $i : X \to \mathbb{P}^N$ such that $L \cong i^*(\mathcal{O}(1))$. In this case by Harshorne Exercise 5.14, the homogeneous coordinate ring of X agrees with the ring $\oplus_{j\geq 0}\Gamma(X, L^{\otimes j})$ for large enough degrees.

(2) $\Longrightarrow$ (3) This is by definition of the Zarisky topology on $Proj(S)$: the basis is formed by open sets $D_+(f) = X - V(f)$ for homogeneous elements $f \in S$.

(3) $\Longrightarrow$ (1) Bondal and Orlov refer to Illusie in SGA6. □

Let $\mathcal{C}$ be a graded category endowed with a Serre functor S. We say that an object $P \in \mathcal{C}$ is a point object of codimension s if

(i) $S(P) \cong P[s]$
(ii) $Hom(P, P[j]) = 0, \ j < 0$
(iii) $Hom(P, P) = \mathbb{C}$.

We denote the set of point objects in $\mathcal{C}$ by $\widetilde{\mathcal{P}}(\mathcal{C})$. It is obvious that for a smooth projective variety X of dimension n the set $\widetilde{\mathcal{P}}(D^b(X))$ contains the set of objects isomorphic to shifts of skyscraper sheaves $k(x)[j]$ for $x \in X$, $j \in \mathbb{Z}$ (all such point objects have codimension n).

We say that an object $L \in \mathcal{C}$ is a line bundle object if for any point object P there exists a $t \in \mathbb{Z}$ such that

(i) $Hom(L, P[t]) = \mathbb{C}$
(ii) $Hom(L, P[j]) = 0$ for $j \neq t$

We denote the set of invertible objects in $\mathcal{C}$ by $\widetilde{\mathcal{L}}(\mathcal{C})$. Note that both $\widetilde{\mathcal{L}}(\mathcal{C})$ and $\widetilde{\mathcal{P}}(\mathcal{C})$ are closed under shifts.

The first step in proving Theorem 3.1 relies on the following statement:

Proposition 3.4. *1. If X is a smooth projective variety with $K_X > 0$ or $K_X < 0$, then the set of point objects $\widetilde{\mathcal{P}}(D^b(X))$ coincides with shifts of skyscraper sheaves.*

2. If $\widetilde{\mathcal{P}}(D^b(X))$ coincides with shifts of skyscraper sheaves, then the set of invertible objects $\widetilde{\mathcal{L}}(D^b(X))$ coincides with shifts of line bundles.

Before we prove the Proposition we prove two Lemmas characterizing skyscraper sheaves and vector bundles.

Lemma 3.5. *If $\mathcal{F}$ is a coherent sheaf on a projective variety X, such that*

$$\mathcal{F} \otimes L \cong \mathcal{F}$$

for some ample line bundle L on X, then $\mathcal{F}$ has a zero-dimensional support.

Proof. We can assume that L is very ample: $L = i^*\mathcal{O}(1)$ for some embedding $i : X \to \mathbb{P}^N$. Let $P(n) = \chi(\mathcal{F}(n))$ be the Hilbert polynomial. It is well known that the degree of $P(n)$ is equal to the dimension of support of $\mathcal{F}$, and since $P(n) = \chi(\mathcal{F})$ is constant by assumption, it follows that $dim(supp(\mathcal{F})) = 0$. □

Lemma 3.6. *If $\mathcal{F}$ is a coherent sheaf on a smooth variety X, such that*

$$Ext^1_X(\mathcal{F}, k(x)) = 0$$

for all $x \in supp(\mathcal{F})$, then $\mathcal{F}$ is locally free.

Proof. We first reduce the statement to the case $X = Spec(A)$, A is a regular local k-algebra by using the adjunction for the flat morphism $j : Spec(\mathcal{O}_{X,x}) \to X$:

$$Ext^1_X(\mathcal{F}, k(x)) = Ext^1_{Spec(\mathcal{O}_{X,x})}(j^*\mathcal{F}, k(x))$$

and then use the following fact form commutative algebra. If A is a regular local ring and M a finitely generated A-module satisfying the property $Ext^1(M, k) = 0$, then M is free. □

Proof of the Proposition. (1) We know that any shift of a skyscraper sheaf on any variety X is a point object. Let us now prove the converse. Let P be a point object in $D^b(X)$ and let $\mathcal{H}^i$ be the cohomology sheaves of P. It follows from the first Lemma above that $\mathcal{H}^i$ have zero-dimensional support and that the codimension of P is equal to n.

Consider the spectral sequence

$$E_2^{p,q} = \bigoplus_{k-j=q} Ext^p(\mathcal{H}^j, \mathcal{H}^k) \implies Hom(P, P[p+q]).$$

Let us consider the smallest q_0 such that E_2^{0,q_0} is non-zero. Since for any non-vanishing $\mathcal{H}^j$ the vector space $Hom(\mathcal{H}^j, \mathcal{H}^j)$ is non-zero, we have $E_2^{0,0} \neq 0$, hence $q_0 \leq 0$. In fact for any $p \in \mathbb{Z}$ and $q < q_0$ we have $E_2^{p,q} = 0$, since non-vanishing of some $Ext^p(\mathcal{H}^j, \mathcal{H}^k)$ implies that $\mathcal{H}^j$ and $\mathcal{H}^k$ have a common point in their supports and hence $Hom(\mathcal{H}^j, \mathcal{H}^k) \neq 0$.

Since the term E_2^{0,q_0} is in the corner of the sheet E_2, that is all terms to the left and below vanish, we have

$$E_2^{0,q_0} \cong E_\infty^{0,q_0} \cong Hom(P, P[q_0]),$$

and $q_0 \geq 0$ by definition of the point object. We have therefore that $q_0 = 0$ and

$$\bigoplus_j Hom(\mathcal{H}^j, \mathcal{H}^j) \cong E_\infty^{0,0} \cong Hom(P, P[0]) = \mathbb{C},$$

and all of the $\mathcal{H}^j$'s but one, vanish and $P[j_0] = \mathcal{H}^{j_0}$ is a skyscraper sheaf.

(2) Let L be an line bundle object in $D^b(X)$ and $\mathcal{H}^i$ be its cohomology sheaves. Consider the spectral sequence

$$E_2^{p,q} = Hom(\mathcal{H}^{-q}, k(x)[p+q]) \implies Hom(L, k(x)[p+q]).$$

Let q_0 be the maximal q such that $\mathcal{H}^q$ is non-zero. Then $E_2^{0,-q_0}$ sits is the last row of E_2 term and therefore

$$E_2^{0,q_0} \cong E_\infty^{0,q_0} \cong Hom(L, k(x)[-q_0])$$
$$E_2^{1,q_0} \cong E_\infty^{1,q_0} \cong Hom(L, k(x)[1-q_0]).$$

We have

$$Hom(L, k(x)[-q_0]) = Hom(\mathcal{H}^{q_0}, k(x)) \neq 0$$

for any x in the support of $\mathcal{H}^{q_0}$, and therefore by definition of the invertible object, $E_2^{1,q_0} = Ext^1(\mathcal{H}^{q_0}, k(x)) = 0$. From the second Lemma above we deduce that in fact $\mathcal{H}^{q_0}$ is a locally free sheaf, in particular all $E_2^{p,q_0} = Ext^p(\mathcal{H}^{q_0}, k(x)) = 0$ except for $p = 0$. The rank of $\mathcal{H}^{q_0}$ is equal to $\dim Hom(\mathcal{H}^{q_0}, k(x)) = 1$, that is $\mathcal{H}^{q_0}$ is a line bundle.

Repeating the above argument with $q < q_0$ we get $\mathcal{H}^q = 0$, hence $L \cong \mathcal{H}^{q_0}[-q_0]$ is isomorphic to a shift of a line bundle.

□

Proof of Theorem 3.1. Assume e.g. that $K_X > 0$.

The proof goes in 4 steps:

(1) Identify $X(k)$ with $X'(k)$ and $Pic(X)$ with $Pic(X')$ (as sets)
(2) Identify Zariski topologies on $X(k)$ and on $X'(k)$
(3) Prove that $K_{X'} > 0$
(4) Prove that $X \cong X'$

(1) Let $\mathcal{P}(X)$ and $\mathcal{L}(X)$ denote the set of objects of $D^b(X)$ isomorphic to skyscraper sheaves and line bundles respectively, and similarly for X'.

By Proposition above we have

$$\widetilde{\mathcal{P}}(X) = \widetilde{\mathcal{P}}(D^b(X)) = \widetilde{\mathcal{P}}(D^b(X')) \supset \widetilde{\mathcal{P}}(X').$$

In fact the last inclusion is also an equality. Indeed, any two objects in $\widetilde{\mathcal{P}}(X)$ are either orthogonal, or differ by a shift. Therefore any object $P \in \widetilde{\mathcal{P}}(D^b(X'))$ which is not in $\widetilde{\mathcal{P}}(X')$ would be orthogonal to all skyscraper sheaves on X', hence it will be zero.

Now it follows from the second claim of the Proposition above that

$$\widetilde{\mathcal{L}}(X) = \widetilde{\mathcal{L}}(D^b(X)) = \widetilde{\mathcal{L}}(D^b(X')) = \widetilde{\mathcal{L}}(X').$$

Fix a line bundle L_0 on X. It corresponds to a shift $L_0'[t]$ of a line bundle L_0' on X'. Adjusting the equivalence of derived categories we may assume that $t = 0$.

Now we have

$$\mathcal{P}(X) = \{P \in \widetilde{\mathcal{P}}(X) : Hom(L_0, P) \neq 0\}$$

and similarly for X', hence $\mathcal{P}(X) = \mathcal{P}(X')$. Furthermore we have

$$\mathcal{L}(X) = \{L \in \widetilde{\mathcal{L}}(X) : Hom(L, P) \neq 0, P \in \mathcal{P}(X)\}$$

and similarly for X', hence $\mathcal{L}(X) = \mathcal{L}(X')$.

Finally:

$$X(k) = \frac{\mathcal{P}(X)}{\simeq} = \frac{\mathcal{P}(X')}{\simeq} = X'(k)$$
$$Pic(X) = \frac{\mathcal{L}(X)}{\simeq} = \frac{\mathcal{L}(X')}{\simeq} = Pic(X').$$

(2) We can recover the Zariski topology on the sets $X(k)$, $X(k)$ as follows. Let $U_\alpha = \{P \in \mathcal{P}(X) : \alpha_P \neq 0\}$, for $\alpha \in Hom(L_1, L_2)$, $L_1, L_2 \in \mathcal{L}(X)$ where α_P is the induced morphism in $Hom(L_2, P) \to Hom(L_1, P)$ and similarly for $X'(k)$.

Each U_α is open in $X(k)$. Moreover since any projective has an ample line bundle (1) $\Longrightarrow$ (3) of Lemma 3.3 implies that there are enough of U_α to form a basis of the Zariski topology.

(3) Since $K_X > 0$ it follows from (1) $\Longrightarrow$ (3) of Lemma 3.3 that there even a smaller basis on $X(k)$, then the one given in (2): we restrict to $\alpha \in Hom(L_0, L_i) = \Gamma(\omega_X^{\otimes i})$, where $L_i = S^i(L_0)[-ni] = L_0 \otimes \omega_X^{\otimes i}$. Thus, Zariski topology on X' admits the base of the same form U_α, $\alpha \in Hom(L'_0, L'_i) = \Gamma(\omega_{X'}^{\otimes i})$ and by (3) $\Longrightarrow$ (1) of Lemma 3.3 we have $K'_X > 0$.

(4) We can recover the pluricanonical rings of X and X':

$$R_X^i := Hom(L_0, L_i) = Hom(L_0, L_0 \otimes \omega_X^{\otimes i}) = \Gamma(X, \omega_X^{\otimes i}).$$

hence $R_X = \oplus R_X^j$ is the pluricanonical ring of X and $X \cong Proj A_X$ by (1) $\Longrightarrow$ (2) of Lemma 3.3.

Finally we have

$$X \cong Proj R_X \cong Proj R_{X'} \cong X'.$$

□