DERIVED CATEGORIES: LECTURE 5

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References

[Ca] Andrei Caldararu, *The Mukai pairing, II: The Hochschild-Konstant-Rosenberg isomorphism*, `arXiv:math/0308080 [math.AG]`

[Gi] Viktor Ginzburg, *Lectures on noncommutative geometry*, `arXiv:math/0506603 [math.AG]`

[Ke] B. Keller, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra 123 (1998), no. 1–3, 223–273.

[Ka1] D.Kaledin, *Homological methods in Non-commutative Geometry*, Lectures in Tokyo 2007–08 `http://imperium.lenin.ru/~kaledin/tokyo/`

[Ka2] D.Kaledin, *Non-commutative Geometry from the homological point of view*, Lectures in Seoul, 2009 `http://imperium.lenin.ru/~kaledin/seoul/`

[Kuz] A. Kuznetsov, *Hochschild homology and semiorthogonal decompositions*, `arXiv:0904.4330v1 [math.AG]`

[Ma] Nikita Markarian, *The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem* `arXiv:math/0610553 [math.AG]`

[We] Weibel, *Introduction to homological algebra*

1. Hochschild homology and cohomology of algebras

Let A be an associative, non necessarily commutative k-algebra, and let $A^e = A \otimes A^{op}$. Exchanging the factors gives rise to an isomorphism $A^e = A \otimes A^{op} \cong A^{op} \otimes A = (A^e)^{op}$. Left A^e-module is the same thing as a right A^e-module and the same thing as A-bimodule, that is a vector space endowed with commuting left and right actions of A.

A can be considered as an A^e-module. Let M be an arbitrary A^e-module. We have the following definitions:

$$\begin{aligned} HH_*(M) &:= Tor_*^{A^e}(A, M) \\ HH^*(M) &:= Ext^*_{A^e}(A, M). \end{aligned} \tag{1.1}$$

To compute these we can use a projective resolution of A as A^e-module. Note that

$$HH_0(M) = Tor_0^{A^e}(A, M) = A \otimes_{A^e} M \cong M/[M, A],$$

in particular

$$HH_0(A) = A/[A, A]$$

and

$$HH^0(M) = Ext^0_{A^e}(A, M) = Hom_{A^e}(A, M) = Z_A(M),$$

in particular

$$HH^0(A) = Z(A).$$

1.1. **Bar complex.** For any two associative k-algebras A and A' we define their free product $A \star A'$ as a vector space generated by finite strings

$$a_1 \star a_1' \star a_2 \star a_2' \star \dots$$

where $a_i \in A$, $a_i' \in A'$. The multiplication is concatenation of strings and we factor out the ideal generated by relations

$$\alpha \star a = a \star \alpha = \alpha a$$

for $\alpha \in k$ and $a \in A \sqcup A'$.

Let $B^\bullet(A)$ be the algebra $A \star k[\epsilon]$. $B^\bullet(A)$ can be endowed with a structure of dg-algebra, that is a multiplicative grading, and a differential of degree $+1$ which respects the multiplication according to the Leibnitz rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^{deg(a)} a \cdot d(b).$$

The grading of $B^\bullet(A)$ is given as

$$deg(\epsilon) = -1$$
$$deg(a) = 0,\ a \in A.$$

The differential is defined on generators by

$$d(\epsilon) = 1$$
$$d(a) = 0,\ a \in A$$

and on general element by the Leibnitz rule.

Explicitly elements of $B^\bullet(A)$ of degree $-k$ have the form $a_1 \epsilon a_2 \epsilon \dots \epsilon a_{k+1}$ (possibly some of $a_i = 1$). We use the "bar" notation $a_1|a_2|\dots|a_{k+1}$ to denote the latter element. It follows that $B^{-k}(A) = A^{\otimes k+1}$ and $d : B^{-k}(A) \to B^{-k+1}(A)$ has the form

$$\begin{aligned} d(a_1|a_2|\dots|a_{k+1}) = {} & a_1a_2|a_3|\dots|a_{k+1} - \\ & - a_1|a_2a_3|\dots|a_{k+1} + \\ & \dots \\ & + (-1)^{k+1} a_1|a_2|\dots|a_k a_{k+1}. \end{aligned}$$

Note that each $B^{-k}(A)$ is in fact an A-bimodule with the action defined as

$$(a \otimes a') \cdot a_1|a_2|\dots|a_{k+1} = aa_1|a_2|\dots|a_{k+1}a'$$

and d is a morphism of A-bimodules. In fact each $B^{-k}(A)$ is a free A-bimodule for $k \geq 1$: indeed, as A-bimodules we have $B^{-k}(A) = A^e \otimes V_k$ for k-vector space $A^{\otimes k-1}$.

Lemma 1.1. *The bar complex $B^\bullet(A)$ is acyclic, that is the complex $(B^{-k}(A), d)_{k \geq 1}$ is a free A^e-resolution of $B^0(A) = A$.*

Proof. For any cycle $\xi \in B^\bullet(A)$ we have

$$\xi = 1 \cdot \xi = d(\epsilon \cdot \xi)$$

therefore ξ is cohomologous to zero. □

The bar resolution is convenient to use for Hochschild homology and cohomology computation.

Lemma 1.2. *Let A be commutative. Then we have the following isomorphisms*

$$HH_1(A) \cong \Omega_A$$
$$HH^1(A) \cong Der(A).$$

Here Ω_A is the module of Kähler differentials

$$\Omega_A = \frac{\{a \cdot db : a, b \in A\}}{\{a \cdot d(b_1 b_2) = ab_2 \cdot d(b_1) + ab_1 \cdot d(b_2)\}}$$

and $Der(A)$ is the module of derivations

$$Der(A) = \{g \in Hom_k(A, A) : g(ab) = ag(b) + bg(a)\}.$$

Proof. We use the bar resolution. Recall that for each $k \geq 1$ we have $B^{-k}(A) = A^e \otimes V_k$ where $V_k = A^{\otimes k-1}$. Therefore for any A-bimodule M we can compute Hochschild homology of M using the complex

$$C_k(M) := B^{-k-1}(A) \otimes_{A^e} M = M \otimes A^{\otimes k},\ k \geq 0 \tag{1.2}$$

$$\cdots \to M \otimes A^{\otimes 2} \to M \otimes A \to M$$

(when the tensor product is taken over the base field we omit the subscript). In particular we have

$$HH_1(A, A) = \frac{Ker(d : A \otimes A \to A)}{Im(d : A \otimes A \otimes A \to A \otimes A)},$$

and it is easy to see using identification (1.2) that for the differential we have

$$d(a \otimes b) = ab - ba = 0$$

$$d(a \otimes b \otimes c) = ab \otimes c - a \otimes bc + ca \otimes b,$$

thus the morphism $a \otimes b \mapsto a \cdot db$ gives an isomorphism $HH_1(A, A) = \Omega_A$.

Similarly we have

$$C^k(M) := Hom_{A^e}(B^{-k-1}(A), M) = Hom(A^{\otimes k}, M),$$

are the terms of the complex computing $HH^*(A, M)$

$$M \to Hom(A, M) \to Hom(A^{\otimes 2}, M) \to \dots$$

and in particular

$$HH^1(A) = \frac{Ker(d : Hom(A, A) \to Hom(A^{\otimes 2}, A))}{Im(d : A \to Hom(A, A))}$$

with differentials

$$d(b)(a) = ab - ba = 0,\ a, b \in A$$

$$d(g)(a_1 \otimes a_2) = a_1 g(a_2) - a_2 g(a_1),\ a_1, a_2 \in A,\ g \in Hom(A, M).$$

Thus $HH^1(A, A)$ coincides with $Der(A)$. $\square$

Remark 1.3. If $X = Spec(A)$ is an affine variety, then

$$HH_1(A) = \Omega_A = \Gamma(X, \Omega^1_X)$$

$$HH^1(A) = Der(A) = \Gamma(X, T_X)$$

In the case X is smooth the Hochschild-Konstant-Rosenberg theorem will give further identification of Hochschild homology with differential forms and of Hochschild cohomology with polyvector fields.

1.2. **Morita invariance.** Two rings A and B are called Morita equivalent if their abelian categories of left modules are equivalent.

Theorem 1.4. *Let P be a (B, A)-bimodule, and denote by F_P the functor $A - mod \to B - mod$ sending M to $P \otimes_A M$. This way we get an equivalence of categories:*

$$(B, A) - bimodules \to Fun(A - mod, B - mod).$$

Here on the right we consider the category of right exact additive functors with morphisms given by natural transformations.

In particular, every functor of this kind is isomorpic to some F_P.

Proof. We only prove essential surjectivity, the fully faithfulness is similar. Let F be a right exact additive functor $A - mod \to B - mod$. Let $P := F(A)$. By functoriality, P is (B, A)-bimodule. The proof goes by constructing a natural transformation $F_P \to F$. Since both functors are right exact and the transformation is an isomorphism on free modules by construction it follows that F_P is isomorphic to F. □

Remark 1.5. Theorem of Toën gives an analog for this in dg-context.

Corollary 1.6. *A and B are Morita equivalent if and only if there exist an (B, A)-bimodule P and a (A, B)-bimodule Q such that $Q \otimes_A P \cong B$ as B-bimodules and $P \otimes_B Q \cong A$ as A-bimodules.*

Example 1.7. Let $A = k$ be a field, V be a finite dimensional k-vector space and $B = End(V)(\cong M_n(k))$. Let $P = V$ considered as $(End(V), k)$-bimodule and $Q = V^*$ considered as $(k, End(V))$-bimodule. We have the following isomorphisms of bimodules:

$$V \otimes_k V^* \cong End(V)$$

$$V^* \otimes_{End(V)} V \cong k.$$

Therefore the conditions of the corollary are satisfied and k and $End(V)$ are Morita equivalent.

Corollary 1.8. *If A and B are Morita equivalent, then $mod - A$ and $mod - B$ are equivalent, $A - mod - A$ and $B - mod - B$ are equivalent, $A - mod_f$ and $B - mod_f$ are equivalent.*

Proposition 1.9. *Let A and B be Morita invariant k-algebras. Then their Hochschild homology, resp. cohomology, are isomorphic.*

Proof. We have an equivalence of categories:

$$F_{P,Q} : (A, A) - bimodules \to (B, B) - bimodules$$

given by the functor

$$M \mapsto P \otimes_A M \otimes_A Q.$$

We have $F_{P,Q}(A) = P \otimes Q \cong B$ as a B-bimodule. Note that $F_{P,Q}$ is a tensor functor. Therefore

$$Tor_*^{(A,A)-bimod}(A, A) \cong Tor_*^{(B,B)-bimod}(B, B)$$

and similarly for cohomology.

□

2. Hochschild homology and cohomology of varieties

Let X be a smooth variety of dimension n and let $\Delta : X \to X \times X$ be the diagonal. We do not assume that X is projective.

Definition 2.1. Hochschild homology and cohomology complexes of X are defined as

$$\mathcal{HH}_\bullet := p_{1*}\big(\mathcal{O}_\Delta \otimes \mathcal{O}_\Delta\big)$$

$$\mathcal{HH}^\bullet := p_{1*}\underline{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

Hochschild homology and cohomology groups of X are

$$HH_k(X) := R\Gamma^{-k}(X, \mathcal{HH}_\bullet) \simeq Tor_k^{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

$$HH^k(X) := R\Gamma^k(X, \mathcal{HH}^\bullet) \simeq Ext^k_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

(in agreement with (1.1) when $X = Spec(A)$).

Proposition 2.2. *We have the following expression for Hochschild homology and cohomology complexes:*

$$\mathcal{HH}_\bullet \simeq \Delta^*\Delta_*\mathcal{O}_X$$

$$\mathcal{HH}^\bullet \simeq \Delta^*\Delta_*\mathcal{O}_X \otimes \omega_X^\vee[-n]$$

and

$$\mathcal{HH}_\bullet = p_{1*}\underline{Hom}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X[n]).$$

Proof. For homology we use the projection formula to compute

$$\mathcal{HH}_\bullet = p_{1*}\big(\mathcal{O}_\Delta \otimes \mathcal{O}_\Delta\big) \simeq p_{1*}\Delta_*\Delta^*\mathcal{O}_\Delta \simeq \Delta^*\mathcal{O}_\Delta = \Delta^*\Delta_*\mathcal{O}_X.$$

For cohomology we first note that by Lemma below we have

$$N_{X/X\times X} = T_X \implies det(N_{X/X\times X}) = \omega_X^\vee$$

$$\mathcal{O}_\Delta^\vee \simeq \Delta_*\omega_X^\vee[-n]$$

and then compute using the projection formula:

$$\begin{aligned} \mathcal{HH}^\bullet = p_{1*}\underline{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) &\simeq p_{1*}\big(\mathcal{O}_\Delta^\vee \otimes \mathcal{O}_\Delta\big) \simeq \\ &\simeq p_{1*}\Delta_*\big(\Delta^*\mathcal{O}_\Delta \otimes \omega_X^\vee[-n]\big) \simeq \\ &\simeq \Delta^*\Delta_*\mathcal{O}_X \otimes \omega_X^\vee[-n]). \end{aligned}$$

□

Lemma 2.3. *Let $i : Y \to X$ be a closed embedding of smooth projective varieties of codimension c and let $\mathcal{F}$ be a coherent sheaf on Y. Then*

1.

$$(i_*\mathcal{F})^\vee \simeq i_*(\mathcal{F}^\vee \otimes \det(N_{Y/X})[-c])$$

where $det(N_{Y/X}^\vee)$ is the the top exterior power of the conormal bundle of Y in X. In particular

$$(i_*\mathcal{O}_Y)^\vee \simeq i_*(\det(N_{Y/X})[-c].$$

2. $i^*i_*\mathcal{F}$ has non-zero cohomology sheaves $\mathcal{H}^k$ only in the range $[-k, 0]$ and they are given as

$$\mathcal{H}^k = \mathcal{F} \otimes \Lambda^k N_{Y/X}^\vee.$$

Proof. Let D be the duality operator

$$D_X := \underline{Hom}(\bullet, \omega_X[dim(X)]) = (\bullet)^\vee \otimes \omega_X[dim(X)].$$

It follows from the Grothendieck-Verdier duality formalism that $D_X i_* = i_* D_Y$. We compute

$$\begin{aligned}(i_*\mathcal{F})^\vee &= D_X(i_*\mathcal{F}) \otimes \omega_X^\vee[-dim(X)] = \\ &= i_* D_Y(\mathcal{F}) \otimes \omega_X^\vee[-dim(X)] = \\ &= i_*(\mathcal{F}^\vee \otimes \omega_Y[dim(Y)]) \otimes \omega_X^\vee[-dim(X)] = \\ &= i_*(\mathcal{F}^\vee \otimes \omega_Y \otimes i^*\omega_X^\vee[-c]) = \\ &= i_*(\mathcal{F}^\vee \otimes \det(N_{Y/X})[-c]).\end{aligned}$$

□

Proposition 2.4. *1. Hochschild homology is functorial for Fourier-Mukai transforms.*
2. Hochschild cohomology is functorial for Fourier-Mukai equivalences.

Proof. We omit the proof. See Section 6 of [Kuz]. □

Theorem 2.5. *If X and Y are derived equivalent, then $HH_*(X) \cong HH_*(Y)$ and $HH^*(X) \cong HH^*(Y)$.*

Proof. By a theorem of Orlov, any equivalence $D^b(X) \to D^b(Y)$ is a Fourier-Mukai transform. The result now follows from the previous Proposition. □

3. Hochschild-Konstant-Rosenberg's theorem

Theorem 3.1 (HKR)**.** *We have the following quasi-isomorphisms of complexes on X:*

$$\mathcal{HH}_\bullet \simeq \bigoplus_{i=0}^{n} \Omega_X^i[i]$$

$$\mathcal{HH}^\bullet \simeq \bigoplus_{i=0}^{n} T_X^i[-i]$$

In this section we give a sketch of the proof of this theorem. We start with the following Lemma:

Lemma 3.2. *Let $Y \subset X$ be a smooth complete intersection of codimension n, or more generally, $Y = Z(s)$ for a regular section s of a vector bundle $\mathcal{E}$ of rank n over X. Then we have*

$$\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \Lambda^*(N^\vee_{Y/X}[1])$$
$$\underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \Lambda^*(N_{Y/X}[-1])$$

(both complexes on the right have zero differentials).

Proof. We have the following locally free Kozsul resolution of $\mathcal{O}_Y$:

$$K^\bullet(s) := [\Lambda^n(\mathcal{E}^\vee) \to \Lambda^{n-1}(\mathcal{E}^\vee) \to \cdots \to \Lambda^2(\mathcal{E}^\vee) \to \mathcal{E}^\vee \to \mathcal{O}_X]$$

with differentials given as contraction with s. Now $\mathcal{E}\big|_Y = N_{Y/X}$ and we have:

$$\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y \cong K^\bullet(s) \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \Lambda^*(\mathcal{E}^\vee\big|_Y[1]) = \Lambda^*(N^\vee_{Y/X}[1])$$
$$\underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \cong Hom_{\mathcal{O}_X}(K^\bullet(s), \mathcal{O}_Y) = \Lambda^*(\mathcal{E}\big|_Y[-1]) = \Lambda^*(N_{Y/X}[-1])$$

and the differentials in both complexes are zero since s vanishes on $Y \subset X$. □

Corollary 3.3. *The HKR theorem holds for a regular local k-algebra A.*

Proof. Let $\mathcal{M} \subset A$ be the maximal ideal of A. Let $B = (A \otimes A)_{m^{-1}(\mathcal{M})}$, this is also a regular local ring, and A is a B-module.

One can see that

$$\mathcal{HH}_\bullet = A \otimes^L_B A$$
$$\mathcal{HH}_\bullet = RHom_B(A, A).$$

Since A and B are regular local rings, the ideal $I = Ker(B \to A)$ is generated by a regular sequence $f_1, \dots, f_n \in B$ of length $n = dim(A) = dim(B) - dim(A)$ and we apply the Lemma above. To finish the proof recall that $N_{X/X\times X} = T_X$ and $N^\vee_{X/X\times X} = \Omega^1_X$. □

Proof of HKR theorem. In fact the two statements are equivalent to each other. This follows from Proposition 2.2 and isomorphisms

$$\Omega^i_X = \Lambda^{n-i} T_X \otimes \omega_X$$

which are easy to see fiberwise. We now prove the isomorphism for Hochschild homology. Note that by definition $\mathcal{HH}_\bullet$ is a commutative algebra object in $D^b(X)$. This allows us to define a morphism $\psi_\bullet : \bigoplus_{i=0}^n \Omega^i_X[i] \to \mathcal{HH}_\bullet$ by giving its component $\psi_1 : \Omega_X[1] \to \mathcal{HH}_\bullet$. □

Corollary 3.4.

$$HH_k(X) \cong \bigoplus_{p-q=-k} H^p(X, \Omega^q_X)$$
$$HH^k(X) \cong \bigoplus_{p+q=k} H^p(X, \Lambda^q T_X)$$

Proof.

$$HH_k(X) = R\Gamma^{-k}(X, \bigoplus_{i=0}^n \Omega^i_X[i]) = \bigoplus_{i=0}^n H^{i-k}(X, \Omega^i_X) = \bigoplus_{q-p=k} H^p(X, \Omega^q_X).$$

$$HH^k(X) = R\Gamma^k(X, \bigoplus_{i=0}^{n} \Lambda^i T_X[-i]) = \bigoplus_{i=0}^{n} H^{-i+k}(X, \Lambda^i T_X) = \bigoplus_{p+q=k} H^p(X, \Lambda^q T_X).$$

□

Remark 3.5. Assume that X is a smooth projective over $\mathbb{C}$, so that $H^*(X, \mathbb{C})$ admits a Hodge decomposition. The Euler characterstic of X is equal to the Euler characteristic of the Hochschild homology:

$$\chi(X) = \sum (-1)^k dim HH_k(X),$$

in particular $\chi(X)$ is derived invariant. It is not known whether the Betti numbers $b_i(X)$ are derived invariant.

4. Hochschild homology and cohomology of admissible subcategories

Let $\mathcal{A} \subset D^b(X)$ be an admissible subcategory. There are two possible ways of defining Hochschild homology and cohomology of $\mathcal{A}$.

On the one hand one can use Bondal-van den Bergh to deduce that $\mathcal{A}$ has a strong generator T, and then by a theorem of Keller we have $\mathcal{A} \cong D^b_{perf}(mod - A)$, where A is the dg-algebra of endomorphisms of T. Then one defines

$$HH_*(\mathcal{A}) := HH_*(A)$$
$$HH^*(\mathcal{A}) := HH^*(A).$$

On the other hand Kuznetsov [Kuz] defines Hochschild homology and cohomology of $\mathcal{A}$ as follows: let $P \in D^b(X \times X)$ be the Fourier-Mukai kernel giving the projection $D^b(X) \to \mathcal{A}$. Then

$$HH_*(\mathcal{A}) := Hom_{D^b(X\times X)}(P, P \circ S_X)$$
$$HH^*(\mathcal{A}) := Hom_{D^b(X\times X)}(P, P).$$

Kuznetsov has shown that the two definitions agree. Moreover we have the following fact ([Kuz], Theorem 7.3):

Proposition 4.1. *If $D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$ is a semi-orthogonal decomposition, then there is a canonical decomposition*

$$HH_*(X) = HH_*(\mathcal{A}_1) \oplus \cdots \oplus HH_*(\mathcal{A}_r).$$