DERIVED CATEGORIES: LECTURE 6

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References

[AO] V. Alexeev, D. Orlov, *Derived categories of Burniat surfaces and exceptional collections*, `arXiv:1208.4348v2`

[BBS] Christian Böhning, Hans-Christian Graf von Bothmer, Pawel Sosna, *On the derived category of the classical Godeaux surface*, `arXiv:1206.1830v1`

[BBKS] Christian Böhning, Hans-Christian Graf von Bothmer, Ludmil Katzarkov, Pawel Sosna: *Determinantal Barlow surfaces and phantom categories*, `arXiv:1210.0343`

[GO] Sergey Gorchinskiy, Dmitri Orlov: *Geometric Phantom Categories*, `arXiv:1209.6183`

[GS] S.Galkin, E.Shinder, *Exceptional collections of line bundles on the Beauville surface*, `arXiv:1210.3339 [math.AG]`

1. The Grothendieck group

Let $\mathcal{A}$ be an abelian category. The Grothendieck group $K_0(\mathcal{A})$ is defined as an abelian group with generators $[A]$ for each isomorphism class $A \in \mathcal{A}$ and relations of the form

$$[A] = [A'] + [A'']$$

for each short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$.

Analogously, if $\mathcal{C}$ is a triangulated category, then the Grothendieck group $K_0(\mathcal{C})$ is defined as an abelian group with generators $[A]$ for each isomorphism class $A \in \mathcal{C}$ and relations of the form

$$[A] = [A'] + [A'']$$

for each distinguished triangle $A' \to A \to A'' \to A'[1]$. Note that it follows that $[A[1]] = -[A]$.

Lemma 1.1. *For an abelian category $\mathcal{A}$ the natural morphism*

$$K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$$

is an isomorphism.

Proof. Let $\phi : K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ be the morphism which assigns $[A]$ to $[A[0]]$. We need to construct the inverse to ϕ.

Let $\psi : Iso(D^b(\mathcal{A})) \to K_0(\mathcal{A})$ be the morphism defined as

$$\psi(A^\bullet) := \sum_p (-1)^p [\mathcal{H}^p(A^\bullet)].$$

For a distinguished triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

we get a long exact sequence

$$\cdots \to \mathcal{H}^p(A^\bullet) \to \mathcal{H}^p(B^\bullet) \to \mathcal{H}^p(C^\bullet) \to \mathcal{H}^{p+1}(A^\bullet) \to \dots,$$

thus

$$\psi(B^\bullet) = \psi(A^\bullet) + \psi(C^\bullet) \in K_0(\mathcal{A}).$$

This shows that ψ descends to a well-defined homomorphism

$$\psi : K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A}).$$

By construction we have $\psi(\phi([A])) = [A]$ for $A \in \mathcal{A}$, so that ϕ is injective. Filtering an object by its terms we also see that ϕ is surjective. Hence ϕ is an isomorphism and $\phi^{-1} = \psi$.

□

Definition 1.2. Let X be a variety. The Grothendieck group $K_0(X)$ is defined as

$$K_0(X) := K_0(Coh(X)) \simeq K_0(D^b(X)).$$

Proposition 1.3. *If $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semi-orthogonal decomposition of triangulated categories, then there is a direct sum decomposition*

$$K_0(\mathcal{C}) = K_0(\mathcal{A}) \oplus K_0(\mathcal{B}).$$

Proof. A triangulated functor between triangulated categories induces a morphism on Grothendieck groups.

Thus the embeddings $\mathcal{A} \subset \mathcal{C}$, $\mathcal{B} \subset \mathcal{C}$ give a canonical morphism $K_0(\mathcal{A}) \oplus K_0(\mathcal{B}) \to K_0(\mathcal{C})$ and its inverse is given by the sum of the morphisms induced by the two projections $\mathcal{C} \to \mathcal{A}$, $\mathcal{C} \to \mathcal{B}$. □

$K_0(X)$ is endowed with the Euler bilinear form

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_p Ext^p(\mathcal{F}, \mathcal{G}). \tag{1.1}$$

One uses the pairing (1.1) to define a *numerically exceptional collection* as a sequence of objects $E_1, \dots, E_r \in D^b(X)$ such that $\chi([E_j], [E_i]) = 0$ for $j > i$. Note that if a sequence is exceptional, then it is also numerically exceptional, but not vice-versa.

2. Phantoms and quasi-phantoms

Let $\mathcal{A} \subset D^b(X)$ be an admissible subcategory. It has been thought that if $HH_*(\mathcal{A}) = 0$ or $K_0(\mathcal{A}) = 0$, then $\mathcal{A} = 0$. Recent constructions [BBS], [AO], [GS], [GO], [BBKS] showed that this is not the case.

Following [GO] we give the definitions:

Definition 2.1. If $\mathcal{A} \neq 0$, $K_0(\mathcal{A}) = 0$, $\mathcal{A}$ is called a phantom.

If $\mathcal{A} \neq 0$, $HH_*(\mathcal{A}) = 0$ and $K_0(\mathcal{A})$ is finite torsion, then $\mathcal{A}$ is called a quasi-phantom.

Lemma 2.2. *Let X be a smooth projective variety with an exceptional collection $E_1, \dots, E_r$. Write $D^b(X) = \langle E_1, \dots, E_r, \mathcal{A} \rangle$. Assume that the sum of all Betti numbers $\sum b_k(X)$ is equal to r. Then $HH_*(\mathcal{A}) = 0$.*

Proof. We have

$$\mathbb{C}^r = H^*(X,\mathbb{C}) = HH_*(X) = HH_*(pt)^r \oplus HH_*(\mathcal{A}) = \mathbb{C}^r \oplus HH_*(\mathcal{A}),$$

therefore $HH_*(\mathcal{A}) = 0$. □

Remark 2.3. The conditions of Lemma 2.2 can be only be satisfied if all cohomology classes on X are of type (p,p). Indeed as shown in the proof of the Lemma we have

$$H^*(X,\mathbb{C}) = HH_*(pt)^r,$$

however $HH_*(pt) = HH_0(pt) = \mathbb{C}$ only can contribute to (p,p) classes. In particular if X is a surface, then $p_g(X) = q(X) = 0$ is required.

3. The Beauville surface and its properties

In what follows G is an abelian group

$$G = (\mathbb{Z}/5)^2 = \mathbb{Z}/5\cdot e_1 \oplus \mathbb{Z}/5\cdot e_2$$

acting on a three dimensional vector space V with induced action on $\mathbb{P}^2 = \mathbb{P}(V)$ given by

$$e_1\cdot(X:Y:Z) = (\zeta_5 X:Y:Z)$$
$$e_2\cdot(X:Y:Z) = (X:\zeta_5 Y:Z),$$

where ζ_5 is the 5-th root of unity. Let C be the plane G-invariant Fermat quintic curve

$$X^5+Y^5+Z^5 = 0.$$

We consider the scheme-theoretic quotient C/G which is isomorphic to $\mathbb{P}^1$ and the quotient map

$$\pi : C \to \mathbb{P}^1$$

of degree 25. Explicitly we may pick coordinates on $\mathbb{P}^1$ such that π is given by the formula

$$\pi(X:Y:Z) = (X^5:Y^5).$$

One easily checks that there are three ramification points on $\mathbb{P}^1$ corresponding to the orbits where G acts non-freely:

$$\begin{aligned} D_1 &= \{(0:-\zeta_5^j:1),\ j=0\dots 4\} \\ D_2 &= \{(-\zeta_5^j:0:1),\ j=0\dots 4\} \\ D_3 &= \{(\zeta_5^j:-\zeta_5^j:0),\ j=0\dots 4\} \end{aligned} \tag{3.1}$$

Stabilizers of the points in $D_i,\ i=1,2,3$ are equal to

$$\begin{aligned} G_1 &= \mathbb{Z}/5\cdot e_1 \\ G_2 &= \mathbb{Z}/5\cdot e_2 \\ G_3 &= \mathbb{Z}/5\cdot(e_1+e_2) \end{aligned} \tag{3.2}$$

respectively.

Lemma 3.1. *The equivariant Picard group $Pic^G(C)$ splits as a direct sum*

$$Pic^G(C) = \widehat{G} \oplus \mathbb{Z} \cdot \mathcal{O}(1).$$

The canonical class is uniquely divisible by 2, and if we write $\mathcal{K}_C(1)$ for the resulting line bundle, $\mathcal{K}_C(1)$ and $\mathcal{O}_C(1)$ differ by torsion, more precisely, we have $\mathcal{K}_C(1) = \mathcal{O}_C(1)[3,3]$.

We introduce the curve C' which is defined by the same equation

$$X^5 + Y^5 + Z^5 = 0$$

as C but has a different G-action. We pick the G-action on C' to be defined as

$$\begin{aligned} e_1 \cdot (X : Y : Z) &= (\zeta_5^2 X : \zeta_5^4 Y : Z) \\ e_2 \cdot (X : Y : Z) &= (\zeta_5 X : \zeta_5^3 Y : Z) \end{aligned}$$

For this action points in divisors $D_i,\ i = 1, 2, 3$ defined as in (3.1) have stabilizers

$$\begin{aligned} G'_1 &= \mathbb{Z}/5 \cdot (e_1 + 2e_2) \\ G'_2 &= \mathbb{Z}/5 \cdot (e_1 + 3e_2) \\ G'_3 &= \mathbb{Z}/5 \cdot (e_1 + 4e_2) \end{aligned} \tag{3.3}$$

respectively.

We let $T = C \times C'$ with the diagonal G-action. Since the stabilizers in (3.2) and (3.3) are distinct, the G-action on T is free. One can check that the corresponding smooth quotient Beauville surface $S = T/G$ is of general type with $p_g = q = 0, K^2 = 8$ (Chapter X, Exercise 4 in [?]). The Noether formula gives $b_2 = 2$. Since $p_g = q = 0$, the exponential exact sequence gives an identification

$$Pic(S) = H^2(S, \mathbb{Z}).$$

Modulo torsion $Pic(S)$ is an indefinite unimodular lattice of rank 2, that is a hyperbolic plane.

We introduce G-linearized line bundles $\mathcal{O}(i, j)$ and $\mathcal{K}(i, j)$ for $i, j \in \mathbb{Z}$ as follows:

$$\mathcal{O}(i, j) = p_1^*(\mathcal{O}(i)) \otimes p_2^*(\mathcal{O}(j))$$
$$\mathcal{K}(i, j) = p_1^*(\mathcal{K}(i)) \otimes p_2^*(\mathcal{K}(j)) = \mathcal{O}(i, j)[3i + 3j, 3i + 2j].$$

Proposition 3.2. *1. The Picard group of S splits as*

$$Pic(S)(= Pic^G(T)) = \widehat{G} \cdot [\mathcal{O}] \oplus \mathbb{Z} \cdot [\mathcal{O}(1, 0)] \oplus \mathbb{Z} \cdot [\mathcal{O}(0, 1)].$$

2. *The Grothendieck group has a decomposition $K_0(S) = \mathbb{Z}^4 \oplus (\mathbb{Z}/5)^2$.*
3. *The canonical class ω_S is equal to $\mathcal{K}(2, 2) = \mathcal{O}(2, 2)[2, 0]$.*
4. *The intersection pairing is given by*

$$(\mathcal{O}(i_1, j_1)(\chi_1) \cdot \mathcal{O}(i_2, j_2)(\chi_2)) = i_1 j_2 + j_1 i_2.$$

5. *The Euler characteristic of a line bundle $L = \mathcal{O}(i, j)(\chi)$ is equal to $(i - 1)(j - 1)$.*

Proof. See [GS], Proposition 2.4 and Lemma 2.7. □

4. Exceptional and numerically exceptional collections on the Beauville surface

Lemma 4.1. *A sequence*

$$\mathcal{O}, L_1, L_2, L_3$$

of line bundles on S is numerically exceptional if and only if it belongs to one of the following four numerical types (that is each object is allowed to be twisted by an arbitrary character):

$$(I_c)\ \mathcal{O}, \mathcal{O}(-1,0), \mathcal{O}(c-1,-1), \mathcal{O}(c-2,-1),\ c \in \mathbb{Z}$$
$$(II_c)\ \mathcal{O}, \mathcal{O}(0,-1), \mathcal{O}(-1,c-1), \mathcal{O}(-1,c-2),\ c \in \mathbb{Z}$$
$$(III_c)\ \mathcal{O}, \mathcal{O}(-1,c), \mathcal{O}(-1,c-1), \mathcal{O}(-2,-1),\ c \in \mathbb{Z}$$
$$(IV_c)\ \mathcal{O}, \mathcal{O}(c,-1), \mathcal{O}(c-1,-1), \mathcal{O}(-1,-2),\ c \in \mathbb{Z}.$$

Proof. It easily follows from 3.2 (5) that all listed sequences are numerically exceptional. For the reverse implication see [GS], Lemma 3.1. □

We now investigate which of the numerically exceptional collections of Lemma 4.1 can be lifted to exceptional collections. Here by a lift we mean a lift with respect to the morphism

$$\mathbb{Z}^2 \oplus \widehat{G} = Pic(S) \to Pic(S)/tors = \mathbb{Z}^2,$$

that is a choice of a character $\chi \in \widehat{G}$. We will need a detailed study of the characters that may appear in the cohomology groups of sheaves on T.

For a G-linearized line bundle on T we define the acyclic set of L as

$$\mathcal{A}(L) := \{\chi \in Hom(G, \mathbb{C}^*)\ :\ \chi \notin [H^*(T,L)]\}$$

By definition $L(\chi)$ is acyclic if and only if $-\chi \in \mathcal{A}(L)$. Since by Proposition 3.2(1), any line bundle on S is isomorphic to some $\mathcal{K}(i,j)(\chi)$, we see from the next lemma that there are 39 isomorphism classes of acyclic line bundles on S.

Lemma 4.2. *The only nonempty acyclic sets of line bundles $\mathcal{K}(i,j)$ on S are:*

$$\begin{aligned}
\mathcal{A}(\mathcal{K}(1,-2)) &= \{[0,0]\}\\
\mathcal{A}(\mathcal{K}(1,-1)) &= \{[0,3],[2,0],[3,2]\}\\
\mathcal{A}(\mathcal{K}(1,0)) &= \{[0,0],[0,1],[0,2],[1,4],[2,3],[3,0],[4,0]\}\\
\mathcal{A}(\mathcal{K}(1,1)) &= \{[0,0],[1,2],[2,1],[2,2],[3,3],[3,4],[4,3]\}\\
\mathcal{A}(\mathcal{K}(1,2)) &= \{[0,0],[0,3],[0,4],[1,0],[2,0],[3,2],[4,1]\}\\
\mathcal{A}(\mathcal{K}(1,3)) &= \{[0,2],[2,3],[3,0]\}\\
\mathcal{A}(\mathcal{K}(1,4)) &= \{[0,0]\}\\
\mathcal{A}(\mathcal{K}(-1,1)) &= \{[0,0]\}\\
\mathcal{A}(\mathcal{K}(0,1)) &= \{[0,0],[3,3],[3,4],[4,3]\}\\
\mathcal{A}(\mathcal{K}(2,1)) &= \{[0,0],[1,2],[2,1],[2,2]\}\\
\mathcal{A}(\mathcal{K}(3,1)) &= \{[0,0]\}.
\end{aligned}$$

Proof. See [GS], Lemma 3.3. □

Theorem 4.3. *The following list contains all exceptional collections of length* 4 *consisting of line bundles on* S *(up to a common twist by a line bundle):*

$$\begin{aligned}
(I_1)\ &\mathcal{O},\ \mathcal{K}(-1,0),\ \mathcal{K}(0,-1),\ \mathcal{K}(-1,-1)\\
(IV_1)\ &\mathcal{O},\ \mathcal{K}(1,-1),\ \mathcal{K}(0,-1),\ \mathcal{K}(-1,-2)\\
(I_{-1})\ &\mathcal{O},\ \mathcal{K}(-1,0),\ \mathcal{K}(-2,-1),\ \mathcal{K}(-3,-1)\\
(IV_{-1})\ &\mathcal{O},\ \mathcal{K}(-1,-1),\ \mathcal{K}(-2,-1),\ \mathcal{K}(-1,-2)\\
(II_0 = IV_0)\ &\mathcal{O},\ \mathcal{K}(0,-1),\ \mathcal{K}(-1,-1),\ \mathcal{K}(-1,-2)\\
(I_0)\ &\mathcal{O},\ \mathcal{K}(-1,0),\ \mathcal{K}(-1,-1),\ \mathcal{K}(-2,-1).
\end{aligned} \tag{4.1}$$

Proof. See [GS], Theorem 3.5. □

Corollary 4.4. *The Beauville surface* S *admits quasi-phantom subcategories* $\mathcal{A}$ *with* $K_0(\mathcal{A}) = (\mathbb{Z}/5)^2$.

Proof. Taking orthogonals to collections in Theorem 4.3 and using Lemma 2.2 we see that $HH_*(\mathcal{A}) = 0$. An argument using additivity of K_0 and Proposition 3.2(2) shows that

$$K_0(\mathcal{A}) = (\mathbb{Z}/5)^2.$$

□