RESEARCH STATEMENT

GROTHENDIECK RING OF VARIETIES AND ITS APPLICATIONS IN BIRATIONAL GEOMETRY

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I am interested in Algebraic Geometry, with my main work concentrating on derived categories of coherent sheaves for singular and nonsingular algebraic varieties, the Grothendieck ring of varieties, and applications of these to algebraic K-theory and birational algebraic geometry.

In this document I explain my program on application of the Grothendieck ring of varieties to birational geometry [1], [2], [3]. The rest of my papers, including my recent work on algebraic K-theory and derived categories of singular algebraic varieties are listed in my CV.

1. Introduction: The Grothendieck ring of varieties. The Grothendieck ring of varieties $K_0(Var/k)$ over a field k is defined to have as its generators isomorphism classes $[X]$ of quasi-projective algebraic varieties, modulo relations

$$[X] = [Z] + [U]$$

where $Z \subset X$ is a closed subvariety with open complement U. Because of these relations, the same group is obtained if instead of quasi-projective varieties one takes for example all schemes of finite type over a field; furthermore if k has characteristic zero, it is sufficient to consider only smooth such schemes as generators.

The ring structure on $K_0(Var/k)$ is induced by the product operation

$$[X] \cdot [Y] = [X \times_k Y],$$

with $1 = [\mathrm{Spec}(k)]$. An important element in $K_0(Var/k)$ is the class of the affine line $\mathbb{L} = [\mathbb{A}^1]$, called the Lefschetz class.

The Grothendieck ring first appeared in the famous Grothendieck-Serre correspondence in the 1960s, as one of the first attempts of Grothendieck to define motives; for this reason the Grothendieck ring is sometimes referred to as the ring of baby motives. Grothendieck ring of varieties, and the analogous ring defined for algebraic stacks is typically used as a book-keeping device in subjects such as motivic integration (Kontsevich, Denef-Loeser) as well as in motivic Hall algebras (Joyce-Song, Bridgeland, Kontsevich-Soibelman).

In what follows below I explain how the Grothendieck ring of varieties can be applied in studying birational geometry.

2. L-equivalence, D-equivalence and rationality of cubics. The following question seems to be among the most central unsolved problems in modern birational geometry:

Question 1. *Are very general smooth cubic fourfolds irrational?*

There has been work with a view towards this question from the Hodge-theoretic perspective by Hassett [Has00], derived categories perspective by Kuznetsov [Kuz10] and Chow groups perspective by Voisin [Vo14] respectively, but so far the problem remains out of reach.

In a joint work with Sergey Galkin we gave a heuristic argument, for why very general smooth cubic fourfold X should be irrational, and furthermore, a heuristic condition has been given for rationality in terms of the Fano variety of lines $F(X)$. The latter condition is compared to the more classical one by Addington [Ad14]. Our heuristics has been based on the properties of the Grothendieck ring of varieties, and has attracted lots of attention from the experts in the field.

Motivated by our work Borisov has shown that the class of the affine line $\mathbb{L} = [\mathbb{A}^1] \in K_0(Var/k)$ is a zero-divisor [Bo14]. To state this result more precisely, it is convenient to use the following definition: we say that two smooth projective varieties X, Y are *L-equivalent* if

$$\mathbb{L}^j \cdot ([X] - [Y]) = 0$$

for some $j \geq 1$. Since the work of Borisov who considered L-equivalent non-isomorphic Calabi-Yau threefolds, many examples have been constructed in the literature.

Among the most exciting examples of L-equivalent varieties are some pairs of D-equivalent (that is having equivalent bounded derived categories of coherent sheaves) non-isomorphic K3 surfaces. These were constructed approximately at the same time independently by Ito, Miura, Okawa and Ueda [IMOU16], Hassett-Lai [HL16] and Kuznetsov and myself [2] (where the term L-equivalence has been introduced). We proved the following result:

Theorem 1 ([2])**.** *Let X and Y be general derived equivalent K3 surfaces of degree* 4 *and* 2 *respectively. Then X and Y are L-equivalent, and in general not isomorphic.*

Our result relies on projective duality between X and Y. By computing the class of a certain moduli space in the Grothendieck ring of varieties in two ways, we obtain the L-equivalence; furthermore we check that on infinitely many Noether-Lefschetz loci in the moduli space the relevant Brauer classes will vanish, while keeping general X and Y not isomorphic. Note that D-equivalence of X and Y in this setup goes back to the work of Mukai. In light of current developments on L-equivalence, the most relevant question seems to be the following one:

Question 2. *What is the geometric meaning of L-equivalence?*

This question has been recently considered by Kawamata [Ka17] (in relation to K-equivalence), Efimov [Ef17] (for abelian surfaces), Huybrechts [Hu17] (for K3 surfaces). On the philosophical note, the conjectural relationship between D-equivalence and L-equivalence [2] can be considered as an enhancement of the classical question whether Hodge numbers are invariant under derived equivalence.

We may make the Question 2 more specific and intriguing by asking:

Question 3. *Do non-isomorphic L-equivalent curves exist?*

Question 4. *Are D-equivalent K3 surfaces also L-equivalent?*

Question 3 admits the following solution:

Theorem 2 (Work in progress, joint with Z. Zhang)**.** *If X, Y are dual quintic genus one curves, then X and Y are L-equivalent and in general non-isomorphic.*

Here by definition the quintic genus one curve is a complete intersection of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with $\mathbb{P}^4 \subset \mathbb{P}^9$, and the dual quintic is defined using projective self-duality of $\mathrm{Gr}(2,5)$. The proof of this Theorem relies on the geometry of $\mathrm{Gr}(2,5)$ and projective duality between the elliptic quintics. This would be the first known example of L-equivalent curves. The subtle point is that in this case X and Y are twisted forms of each other, so over an algebraically closed field X and Y are always isomorphic, and the theorem only gives examples over more general fields.

Over algebraically closed fields Question 3 remains open; my PhD student George Moulantzikos (started 2018) is going to work on this question. It is easy to see that L-equivalent smooth projective curves will have isomorphic non-polarized Jacobians. Since by Torelli theorem the *polarized* Jacobian does determine the curve up to isomorphism, the question has to do with framing known counterexamples to nonpolarized Torelli theorem, e.g. in genus $g = 2$ in terms of projective duality, which then may lead to L-equivalence of such curves.

When attempting Question 4 the natural approach is to collect more examples of L-equivalence, which is what currently happens in the community; recent of work by Kapustka-Rampazzo, Ottem-Rennemo, Manivel, Kapustka-Kapustka-Moschetti, Borisov-Caldararu-Perry all give new examples of pairs of higher dimensional varieties that are D-equivalent and L-equivalent.

Of particular interest are elliptic K3s whose geometry is strongly connected to the geometry of genus one curves over arbitrary fields. For instance from Theorem 2 above we deduce the following:

Theorem 3 (Work in progress, joint with Z. Zhang)**.** *If X is an elliptic K3 surface with a multi-section of degree 5, and $Y = \mathrm{Jac}^2(X)$, the relative second Jacobian, then X and Y are L-equivalent and in general non-isomorphic.*

Here D-equivalence of X and Y follows from the work of Bridgeland.

Finally, completing the circle and returning to the geometry of cubics, I consider the following question to be the most important in all my research so far:

Question 5. *Using the language of L-equivalence, can the heuristics of* [1] *be made into a* proof *that very general cubic fourfolds are irrational?*

A little step to a positive answer to this question is the following restatement of heuristical argument of [1] using L-equivalence.

Theorem 4 (Work in progress)**.** *If the subgroup*

$$\{\alpha \in K_0(Var/\mathbb{C}) : \mathbb{L}^j \cdot \alpha = 0 \textit{ for some } j \geq 1\} \subset K_0(Var/\mathbb{C})$$

is generated by differences $[X] - [Y]$ of smooth projective L-equivalent varieties, then very general complex cubic fourfold is irrational.

The plausibility of the assumption of the Theorem comes from the fact that all known elements in the subgroup in question are constructed using L-equivalence. Proving that this assumption is correct may be very difficult, one possible approach is to rely on birational cobordism of Wlodarczyk [Wl03], and this requires a deep investigation.

3. Specialization and variation for stable birational types. For another application of the Grothendieck ring of varieties in birational geometry, we recall that it is an old and natural question in Algebraic Geometry what is the behaviour of rationality and related notions such as stable rationality, unirationality etc in families. In this context, the most natural question is that of specialization:

Question 6. *Given a nice family $\pi : \mathcal{X} \to S$ of algebraic varieties such that very general fibers are rational, are all fibers rational? In other words, are limits of rational varieties rational?*

These kind of questions attracted particular attention after the development of the Chow group specialization method by Voisin [Vo15], Colliot-Thélène and Pirutka [CTP16], Totaro [To16b] which since then has been used to show that very general members of large classes of algebraic varieties to be stably irrational.

It is clear from the simplest examples that in general Question 1 can not have a positive answer, if the singularities of the fibers are not restricted. However, the question has been open for many years even for smooth families; it appears in particular in Kollar's book [Ko96]. In 2017-18 in a joint work with J. Nicaise we proved the following result:

Theorem 5 ([3]). *If C is a smooth curve over an algebraically closed field of characteristic zero and $\mathcal{X} \to C$ is a proper smooth or nodal family, and very general fibers are stably rational, then all fibers are stably rational.*

Our method relies on passing to the geometric point $\overline{k(C)}$, and using an appropriate specialization for the Grothendieck ring of varieties, as well as the Larsen-Lunts Theorem providing the connection between stable rationality and classes in the Grothendieck ring. Kontsevich and Tschinkel [KT17] used a variation of our method to show that the same is true for rationality rather than stable rationality.

I am convinced that the full strength of the application of Grothendieck ring to stable rationality questions is not attained yet. For example, the Grothendieck ring does not just distinguish stably irrational varieties from stably rational ones, but distinguishes *stable birational types*. Furthermore, the kind of specialization techniques we have allow one to conclude that certain "jumping" behaviour of stable birational types is not possible. More precisely, under some conditions on the family $\pi : \mathcal{X} \to C$ as in the Theorem above, one can show that exactly one of the following must be true:

(1) *Constant stable birational type.* All fibers $\pi^{-1}(c)$, $c \in C$ have the same stable birational type.
(2) *Variation of stable birational type.* Very general fibers $\pi^{-1}(c)$, $\pi^{-1}(c')$, $c, c' \in C$, are not stably birational.

As an application of these ideas, my work in progress is to prove the following very concrete and important result:

Theorem 6 (Work in progress). *For every $n \geq 3$ and $d \geq \log_2(n) + 2$, smooth projective hypersurfaces of degree d and dimension n have varying stable birational type, that is two very general such hypersurfaces are not stably birational to each other.*

My approach to proving the result above is to find a singular but sufficiently nice stably rational hypersurface of given dimension and degree, and given that very general such hypersurfaces are stably irrational [Sch18] (this is where the bound on d comes from), deduce that family can not be stably birationally constant, hence according to the dichotomy explained above, the stable birational types will vary.

The theorem above would be new even for three-dimensional quartic hypersurfaces. Such results are not likely to be obtained by any other existing method, simply because the usual cohomology invariants which are used to prove stable irrationality, such as Chow groups do not distinguish stable birational types.

References

[1] S. Galkin, E. Shinder: *The Fano variety of lines and rationality problem for a cubic hypersurface*, arXiv:1405.5154.

[2] A. Kuznetsov, E. Shinder: *Grothendieck ring of varieties, D- and L-equivalence, and families of quadrics*, arXiv:1612.07193, accepted to Selecta Math.

[3] J. Nicaise, E. Shinder: *The motivic nearby fiber and degeneration of stable rationality*, arXiv:1708.02790.

[Ad14] N. Addington: *On two rationality conjectures for cubic fourfolds*, arXiv:1405.4902.
[Bo14] L. Borisov: Class of the affine line is a zero divisor in the Grothendieck ring. Preprint, arXiv:1412.6194.
[CTP16] J-L. Colliot-Thélène and A. Pirutka: Hypersurfaces quartiques de dimension 3: non-rationalité stable. *Ann. Sci. Éc. Norm. Sup. (4)* 49(2)371–397, 2016.
[Ef17] A. Efimov: *Some remarks on L-equivalence of algebraic varieties.* Selecta Math. (N.S.) 24 (2018), no. 4, 3753–3762.
[Has00] B. Hassett: *Special cubic fourfolds*, Compositio Math. 120 (2000), no. 1, 1–23.
[HL16] B. Hassett, K-W. Lai: *Cremona transformations and derived equivalences of K3 surfaces* arXiv:1612.07751
[Hu17] D. Huybrechts: *Motives of derived equivalent K3 surfaces*, Abh. Math. Semin. Univ. Hambg. 88 (2018), no. 1, 201–207.
[IMOU16] A. Ito, M. Miura, S. Okawa, K. Ueda: *The class of the affine line is a zero divisor in the Grothendieck ring: via K3 surfaces of degree 12*, arXiv:1612.08497.
[Ka17] Yu. Kawamata: *Birational geometry and derived categories* arXiv:1710.07370.
[Ko96] J. Kollár: *Rational curves on algebraic varieties.* Springer-Verlag, Berlin, 1996.
[KT17] M. Kontsevich, Yu. Tschinkel: *Specialization of birational types*, arXiv:1708.05699.
[Kuz10] A. Kuznetsov: *Derived categories of cubic fourfolds*, Cohomological and geometric approaches to rationality problems, 219–243, Progr. Math., 282, Birkhuser Boston, Inc., Boston, MA, 2010.
[LL03] M. Larsen and V. Lunts: Motivic measures and stable birational geometry. *Moscow Math. J.*, 3(1):85–95, 2003.
[Sch18] S. Schreieder: *Stably irrational hypersurfaces of small slopes*, arXiv:1801.05397.
[To16a] B. Totaro: Hypersurfaces that are not stably rational. *J. Amer. Math. Soc.* 29(3):883–891, 2016.
[To16b] B. Totaro: Rationality does not specialise among terminal varieties. *Math. Proc. Cambridge Philos. Soc.* 161(1):13–15, 2016.
[Vo14] C. Voisin: *On the universal CH_0 group of cubic hypersurfaces*, Journal of the EMS, Vol. 19, 6, (2017), 1619–1653.
[Vo15] C. Voisin: Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.* 201(1):207–237, 2015.
[Wl03] J. Wlodarczyk: Toroidal varieties and the weak factorization theorem. *Invent. Math.* 154(2):223–331, 2003.