Introduction to the geometry of holomorphic symplectic manifolds

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4 lectures

The lectures will cover some basic aspects of holomorphic symplectic geometry. By a holomorphic symplectic manifold we will mean a compact simply-connected Kähler manifold carrying a holomorphic symplectic form, that is a holomorphic 2-form which is non-degenerate at every point. One may consider such manifolds as natural higher-dimensional analogues of K3 surfaces, and K3 surfaces are the only holomorphic symplectic manifolds of dimension two. The other examples include Hilbert schemes of points on K3 surfaces and generalized Kummer varieties. There are also two exceptional examples in dimension 6 and 10, and no more examples are known at the moment. The recent progress in the study of holomorphic symplectic geometry was largely influenced by the global Torelli theorem proved by Verbitsky.

The theory of holomorphic symplectic manifolds is essentially based on the fact that such manifolds have nice deformation theory. I will recall the necessary results from deformation theory of Kähler manifolds (existence of universal deformations, Bogomolov-Tian-Todorov theorem) and deduce local Torelli theorem for holomorphic symplectic manifolds.

One of the most important properties of holomorphic symplectic manifolds is the existence of a quadratic form on the second cohomology which is called the Beauville-Bogomolov form. We will discuss in detail where this form comes from and then prove the Fujiki identities for it. From the Fujiki identities one can deduce a lot about the geometry of holomorphic symplectic manifolds. We will discuss some results about the cohomology algebra of such manifolds and if time permits we will prove a basic version of Matsushita's theorem on the structure of holomorphic Lagrangian fibrations.

Another important concept connected to holomorphic symplectic manifolds is the concept of a twistor family. This is a special family of complex structures on the manifold, related to the algebra of quaternions. The existence of such families follows from Calabi-Yau theorem which also provides a link between algebraic and differential-geometric viewpoints on holomophic symplectic manifolds. The existence if twistor families is crucial for the proof of global Torelli theorem (and many other results which we will not have time to discuss). I will try to give an outline of the proof of this beautiful theorem.

Lecture 1. Definition of an irreducible holomorphic symplectic manifold (IHSM). Examples: K3 surfaces, Hilbert schemes of points on K3 surfaces, generalized Kummer varieties. Reminder about deformation theory: Kuranishi theorem, Bogomolov-Tian-Todorov theorem. Period map, period domain, local Torelli theorem.

Lecture 2. Beauville-Bogomolov (BB) form and Fujiki relations. Properties of BB form: it is rational (after rescaling), of signature $(3, b_2(X) - 3)$. Applications: structure of the subring in the cohomology ring of X generated by $H^2(X, \mathbb{C})$, properties of Lagrangian fibrations.

Lecture 3. Hyperkähler metrics and differential-geometric definition of IHSM, the equivalence of two definitions. Twistor families and twistor lines in the moduli space.

Lecture 4. Moduli spaces of marked IHSM's. Global Torelli theorem.