Fano variety of lines and rationality problem for a cubic hypersurface

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What we know and what we’d like to know about cubic hypersurfaces

The Grothendieck ring of varieties and realizations

The $Y-F(Y)$ relation

Applications to rationality of a cubic
Cubic hypersurfaces

- $k$: arbitrary field
- $Y \subset \mathbb{P}^{d+1}$: smooth cubic hypersurface of dimension $d$

Question

Are smooth cubics $Y$ rational varieties (i.e. $k(Y) \simeq k(t_1, \ldots, t_d)$)?
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**Remarks**

- The answer is not known for $d \geq 4$.
- Singular cubics are always rational: projecting from a singular point gives a birational map $Y \dashrightarrow \mathbb{P}^d$.
- If $d \geq 2$, and $k$ is algebraically closed, then smooth cubics $Y$ are unirational varieties: i.e. $k(Y)$ is a finite index subfield of $k(t_1, \ldots, t_d)$.
Rationality of $d$-dimensional cubics

- $d = 1, 2$: trivial (elliptic curve irrational, cubic surface rational over $\overline{k}$)
- $d = 3$: cubic threefold - irrational. This is a very subtle result by Clemens-Griffiths (char. 0) and Murre (char. p)
- $d = 4$: cubic fourfold - unknown. There are divisors in the moduli space parametrizing rational ones [Tregub, Beauville-Donagi, Zarhin, Hassett]. Very general cubic fourfold: irrational??

Constructions of rational cubics in any even dimension $d = 2r$

- Cubics $Y \subset P^{2r+1}$ containing two disjoint $r$-planes $P_1, P_2$: birational morphism $P_1 \times P_2 \to Y$ maps $(a, b) \in P_1 \times P_2$ to the third intersection point of $\langle a, b \rangle$ with $Y$.

- Cubics $Y \subset P^{2r+1}$ containing $r$-dimensional rational scrolls with one apparent double point $W \subset P^r$: there is a birational morphism $\text{Sym}^2(W) \to Y$. Such scrolls exist for any $r$ [Severi, Edge, Russo].
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Fano variety of lines

Definition

Associated to a cubic hypersurface $Y \subset \mathbb{P}^{d+1}$ there is its Fano variety of lines:

$$F(Y) := \{ L \in Gr(1, \mathbb{P}^{d+1}) : L \subset Y \} \subset Gr(1, \mathbb{P}^{d+1})$$
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**Properties of $F(Y)$** [Fano, Altman-Kleiman, Barth-Van de Ven]

- $F(Y) \neq \emptyset$ for $d \geq 2$
- $F(Y)$ is connected for $d \geq 3$
- $Y$ smooth $\implies F(Y)$ smooth projective of dimension $2(d - 2)$
Fano varieties of lines on a $d$-dimensional cubic $Y$

- $d = 1, 2$: no lines on an elliptic curve; 27 lines on a cubic surface over $\kbar$.

- $d = 3$: cubic threefold - $F(Y)$ is a surface of general type with $H^1(F(Y), \mathbb{Z}) \simeq H^3(Y, \mathbb{Z})$ [Fano, Clemens-Griffiths]

- $d = 4$: cubic fourfold - $F(Y)$ is an irreducible holomorphic symplectic 4-fold of $K3$ type and $H^2(F(Y), \mathbb{Q}) \simeq H^4(Y, \mathbb{Q})$ [Beauville-Donagi]

- $d \geq 5$: $F(Y)$: $2(d - 2)$-dimensional variety with $-K_{F(Y)} > 0$. 

The Fano variety appears naturally in all geometric studies of cubic hypersurfaces.
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Question

What is the relation between the geometry of $F(Y)$ and the geometry of $Y$?
The $Y - F(Y)$ relation

joint work with S. Galkin (HSE, Moscow)

- Find relations between $F(Y)$ and $Y$ in the Grothendieck ring of varieties $K(Var/k)$ and in $K(Var/k)[L^{-1}]$

- Use these relations to compute invariants of $F(Y)$, e.g. the Hodge structure

- Deduce criteria for rationality of $Y$ modulo a general motivic conjecture
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Applications to rationality of a cubic
Let $k$ be an arbitrary field. $K(Var/k)$ is:

- **Generators:** $[X], X/k$ quasi-projective variety
- **Relations:** $[X] = [Z] + [U]$ for any closed $Z \subset X$ with open complement $U$
- **Product:** $[X] \cdot [Y] = [X \times Y]$

$L := [A_1]$ - the Lefschetz class (sometimes called the Tate class)

$[P_n] = \sum_{\text{nk}=0} [A_k] = \sum_{\text{nk}=0} L_k$

We don't know: is $L \in K(Var/k)$ a zero-divisor? Still, often one inverts $L$ and considers $K(Var/k)[L^{-1}](\text{motivic integration, motivic Hall algebras})$. 
The Grothendieck ring $K(Var/k)$

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- $[\mathbb{P}^n] = \sum_{k=0}^{n} [\mathbb{A}^k] = \sum_{k=0}^{n} \mathbb{L}^k$

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Properties of the Grothendieck ring

- If $X \to B$ a Zarisky locally trivial fibration with fiber $F$, then
  \[ [X] = [F] \cdot [B] \]

- If $Y = Bl_Z(X)$ is a blow up of a smooth subvariety $Z$ in a smooth variety $X$, then
  \[ [Y] - [\mathbb{P}(N_{Z/X})] = [X] - [Z] \]
Symmetric powers

- For $X/k$ and $n \geq 0$ we consider the space $Sym^n(X) = X^n/\Sigma_n$.

- By definition $Sym^n(X)$ is the space parametrizing unordered $n$-tuples of points on $X$.

- $Sym^n(X)$ are singular if $\dim(X) \geq 2$; partial desingularization: $Hilb_n(X) \rightarrow Sym^n(X)$ ($Hilb_n(X)$ parametrizes length $n$ subschemes of $X$).
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Symmetric operations on $K(Var/k)$

The operations $X \mapsto Sym^n(X)$ descend to $K(Var/k)$ and satisfy:

$$Sym^n(\alpha + \beta) = \sum_{i+j=n} Sym^i \alpha \cdot Sym^j \beta; \quad \alpha, \beta \in K(Var/k)$$
Realizations $K(Var/k) \to R$

A realization is a ring homomorphism $\mu : K(Var/k) \to R$.

Equivalently, a realization is a function $\mu : Var/k \to R$ satisfying

1. $\mu(X) = \mu(Z) + \mu(U)$ for each closed $Z \subset X$ with complement $U$
2. $\mu(X \times Y) = \mu(X) \cdot \mu(Y)$
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Examples of realizations

▶ Counting points over a finite field $k = \mathbb{F}_q$: $R = \mathbb{Z}$, $# : X \mapsto #X(\mathbb{F}_q)$.

▶ Topological Euler characteristic for $k = \mathbb{R}$ or $k = \mathbb{C}$: $R = \mathbb{Z}$, $\chi : X \mapsto \chi_c(X(k))$.

▶ Hodge realization for $k = \mathbb{C}$: $R = K(HS/\mathbb{Q})$ (the Grothendieck ring of rational pure polarizable Hodge structures), $\mu_{Hdg} : X \mapsto gr_W H^*(X, \mathbb{Q})$. 
The Grothendieck ring and birational geometry

Unfortunately, we don’t know:

▶ Does \([X] = [Y]\) imply that \(X\) and \(Y\) admit stratifications \(\{X_i\}, \{Y_j\}\) with \(X_i \simeq Y_i\) after renumbering?

▶ Even weaker: Does \([X] = [Y]\) imply that \(X\) and \(Y\) are birational?

▶ These questions are related to hard cancellation conjectures in algebraic geometry.

Some good news (the theorem of Larsen-Lunts)

\([X] = [Y] \implies X, Y\) stably birational.

This means: \(X \times \mathbb{P}^k\) is birational to \(Y \times \mathbb{P}^l\) for some \(k,l\).

For varieties of non-negative Kodaira dimension, this is the same as birational.

▶ More precisely: let \(k\) be a field of characteristic zero. If \(X\) and \(Y_1, \ldots, Y_m\) are smooth projective irreducible varieties and \([X] \equiv \sum_{j=1}^{m} n_j [Y_j] \pmod{L}\), for some \(n_j \in \mathbb{Z}\), then \(X\) is stably birational to one of the \(Y_j\).
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Expressing $Sym^2(Y)$ in terms of $F(Y)$, $Y$ in $K(Var/k)$

Theorem

Let $Y/k$ be a (possibly singular) cubic hypersurface of dimension $d$. We have the following relations in $K(Var/k)$:

1. $[Hilb_2(Y)] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F(Y)]$

2. $[Sym^2(Y)] = (1 + \mathbb{L}^d)[Y] + \mathbb{L}^2[F(Y)] - \mathbb{L}^d[Sing(Y)].$
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Immediate consequences

- Number of lines on real and complex cubic surfaces, smooth or singular.
- E.g. a smooth real cubic surface has 3, 7, 15 or 27 real lines (known since Schl"afli 1863!)
- Euler characteristic of $\chi(F(Y))$, number of points of $F(Y)$ over finite fields, number of components of $Y$ over $\mathbb{R}$ etc.
Idea of the proof

▶ We prove $[\text{Hilb}_2(Y)] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F(Y)]$, the version with $\text{Sym}^2(Y)$ follows easily.
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- Alternatively: two general points on $y_1, y_2 \in Y$ determine a line $L = L_{y_1,y_2}$ together with the third intersection point $y \in Y \cap L$. 
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- In other words, we constructed a birational isomorphism

\[
\phi : Hilb_2(Y) \longrightarrow Fl(Y) = \{(y \in Y \cap L, L \subset \mathbb{P}^{d+1})\}
\]

\(Fl(Y) \rightarrow Y\) is a \(\mathbb{P}^d\)-bundle, hence \([Hilb_2(Y)] = [\mathbb{P}^d][Y] + \text{extra terms}.\)
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- We prove $[\text{Hilb}_2(Y)] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F(Y)]$, the version with $\text{Sym}^2(Y)$ follows easily.

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- The extra terms come from the indeterminacy loci of $\phi$ and $\phi^{-1}$. Both are bundles over $F(Y)$. This gives the desired relation.
The class $\mathcal{M}_Y$ of a $d$-dimensional cubic $Y$ ($d \geq 2$)

- Our next goal is to express $[F(Y)]$ in terms of $[Y]$. This is only possible in $K(\text{Var}/k)[\mathbb{L}^{-1}]$.

- We introduce an auxiliary class $\mathcal{M}_Y := \frac{[Y] - [\mathbb{P}^d]}{\mathbb{L}} \in K_0(\text{Var}/k)[\mathbb{L}^{-1}]$:

$$[Y] = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_Y \in K_0(\text{Var}/k)[\mathbb{L}^{-1}] .$$

$\mathcal{M}_Y$ carries the information about the interesting part of $[Y]$. 

The intuition comes from Hodge theory. Weak Lefschetz:

$$H^*(Y, \mathbb{Q}) = H^*(\mathbb{P}^d, \mathbb{Q}) \oplus H^d(Y, \mathbb{Q}) \text{prim}.$$ 

We have $H^Y = \mu \cdot H^d(M_Y)$, where $H^Y = H^d(Y, \mathbb{Q}) \text{prim}(1)$. 

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The $Y-F(Y)$ relation revisited

Expressing $F(Y)$ in terms of $M_Y$ in $K(Var/k)[L^{-1}]$

**Theorem**

Let $M_Y = \frac{[Y]-[\mathbb{P}^d]}{L} \in K_0(Var/k)[L^{-1}]$. Then:

$$[F(Y)] = Sym^2(M_Y + [\mathbb{P}^{d-2}]) - L^{d-2} \in K_0(Var/k)[L^{-1}]$$
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**Theorem**

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**Corollary**

If $Y$ is a smooth cubic $d$-fold over $\mathbb{C}$. Let $\mathcal{H}_Y = H^d(Y, \mathbb{Q})^\text{prim}(1)$. Then:

$$H^*(F(Y), \mathbb{Q}) \simeq \text{Sym}_2(\mathcal{H}_Y) \oplus \bigoplus_{k=0}^{d-2} \mathcal{H}_Y(-k) \oplus \bigoplus_{k=0}^{2d-4} \mathbb{Q}(-k)^{a_k}$$

where $a_k \in \mathbb{Z}$ are given by an explicit formula.

In particular: $\mathcal{H}_Y \subset H^{d-2}(F(Y), \mathbb{Q})$ and $\text{Sym}_2(\mathcal{H}_Y) \subset H^{2(d-2)}(F(Y), \mathbb{Q})$. 
Hodge structure of $F(Y): Y/\mathbb{C}$ smooth cubic threefold

- $H^3(Y, \mathbb{Q})$: weight 3 and Hodge numbers $(0, 5, 5, 0)$
- $\mathcal{H}_Y$: weight 1 and Hodge numbers $(5, 5)$
- $F(Y)$ smooth projective surface of general type with

\[
H^*(F(Y), \mathbb{Q}) = \begin{bmatrix}
H^4 & 1 \\
H^3 & 5 & 5 \\
H^2 & 10 & 25 & 10 \\
H^1 & 5 & 5 \\
H^0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\mathbb{Q}(-2) \\
\mathcal{H}_Y(-1) \\
\text{Sym}^2(\mathcal{H}_Y) \\
\mathcal{H}_Y \\
\mathbb{Q}
\end{bmatrix}
\]

- This was known before [Clemens-Griffiths]
Hodge structure of $F(Y)$: $Y/\mathbb{C}$ smooth cubic fourfold

- $H^4(Y, \mathbb{Q})$: weight 4 and Hodge numbers $(0, 1, 21, 1, 0)$,
- $\mathcal{H}_Y$ has weight 2 and Hodge numbers $(1, 20, 1)$
- $F(Y)$ irreducible holomorphic symplectic fourfold with $H^*(F(Y), \mathbb{Q})$:

$$
\begin{bmatrix}
H^8 & 1 & 0 \\
H^6 & 1 & 21 & 1 \\
H^4 & 1 & 21 & 232 & 21 & 1 \\
H^2 & 1 & 21 & 1 \\
H^0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbb{Q}(-4) \\
\mathcal{H}_Y(-2) \oplus \mathbb{Q}(-3) \\
\text{Sym}^2(\mathcal{H}_Y) \oplus \mathcal{H}_Y(-1) \oplus \mathbb{Q}(-2) \\
\mathcal{H}_Y \oplus \mathbb{Q}(-1) \\
\mathbb{Q}
\end{bmatrix}
$$

and $H^{2p+1}(F(Y), \mathbb{Q}) = 0$.

- Beauville-Donagi deduced this from the fact that $F(Y)$ is deformation equivalent to $\text{Hilb}_2(K3)$. 
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The Weak Factorization Theorem

$k$: a field of characteristic zero.

Theorem (Weak Factorization Theorem – Wlodarczyk et al)

If $X$ and $Y$ are smooth projective birational varieties, then $Y$ can be obtained from $X$ using a finite number of blow ups and blow downs with smooth centers.
The Weak Factorization Theorem

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If $X$ and $Y$ are smooth projective birational varieties, then $Y$ can be obtained from $X$ using a finite number of blow ups and blow downs with smooth centers.

Corollary

If $X$ is a rational smooth projective $d$-dimensional variety, then

\[ [X] = [\mathbb{P}^d] + L \cdot M_X \]

where $M_X$ is a linear combination of classes of smooth projective varieties of dimension $d - 2$. 
Criterion for irrationality of a cubic in any dimension

The Cancellation Conjecture

- $L$ is not a zero-divisor in $K(Var/k)$ (at the moment this is not known for any field).
Criterion for irrationality of a cubic in any dimension

The Cancellation Conjecture

- $L$ is not a zero-divisor in $K(Var/k)$ (at the moment this is not known for any field).

Theorem

Let $Y$ be a smooth rational $d$-dimensional cubic ($d \geq 3$) over a field satisfying the Cancellation Conjecture.

Then $F(Y)$ is stably birationally equivalent to one of the

$$\text{Hilb}_2(V) \text{ or } V \times V'$$

where $V, V'$ are smooth projective $(d - 2)$-dimensional varieties.
Steps of the proof

▶ Weak Factorization: $Y$ rational $\implies Y = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_Y$ for

$$\mathcal{M}_Y = \sum_{i=1}^{n} [V_i] - \sum_{j=1}^{m} [W_j], \quad \dim V_i = \dim W_j = d - 2.$$
Steps of the proof

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$$
\mathcal{M}_Y = \sum_{i=1}^{n} [V_i] - \sum_{j=1}^{m} [W_j], \quad \dim V_i = \dim W_j = d - 2.
$$

- Recall the $Y-F(Y)$ relation:

$$
[F(Y)] = Sym^2(\mathcal{M}_Y + [\mathbb{P}^{d-2}]) - \mathbb{L}^{d-2} \in K(Var/k)[\mathbb{L}^{-1}]
$$

If $\mathbb{L}$ is not a zero-divisor, this in fact holds in $K(Var/k)$!
Steps of the proof

▶ Weak Factorization: $Y$ rational $\implies Y = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_Y$ for

$$\mathcal{M}_Y = \sum_{i=1}^n [V_i] - \sum_{j=1}^m [W_j], \quad \dim V_i = \dim W_j = d - 2.$$ 

▶ Recall the $Y$-$F(Y)$ relation:

$$[F(Y)] = \text{Sym}^2(\mathcal{M}_Y + [\mathbb{P}^{d-2}]) - \mathbb{L}^{d-2} \in K(\text{Var}/k)[\mathbb{L}^{-1}]$$

If $\mathbb{L}$ is not a zero-divisor, this in fact holds in $K(\text{Var}/k)$!

▶ Reduce mod $\mathbb{L}$ and compute:

$$[F(Y)] \equiv \text{Sym}^2(\mathcal{M}_Y + \mathbb{P}^{d-2}) \pmod{\mathbb{L}}$$

$$\in \langle [V \times V'], [\text{Hilb}_2 V] \rangle \pmod{\mathbb{L}}$$
Steps of the proof

▶ Weak Factorization: \( Y \) rational \( \implies Y = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_Y \) for

\[
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▶ [Larsen-Lunts] \( \implies F(Y) \) is stably birational to one of the \( V \times V' \) or \( \text{Hilb}_2 V \).
Irrationality of cubic threefolds

Theorem

Over a field $k$ satisfying the Cancellation Conjecture smooth cubic threefolds $Y$ are irrational.
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Over a field \( k \) satisfying the Cancellation Conjecture smooth cubic threefolds \( Y \) are irrational.

Proof.

- \( F(Y) \) stably birational to \( C \times C' \) or \( Sym^2(C') \) \( \implies \) they are birational (as \( F(Y) \) is of general type)
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- Impossible! $Sym^2(C)$ is a minimal model, but

\[ 25 = h^{1,1}(F(Y)) < h^{1,1}(Sym^2(C)) = 26. \]
(Ir)-rationality of cubic fourfolds

- In the moduli of all cubic fourfolds there are countably many divisors which parametrize rational ones [Tregub, Zarhin, Hassett]

- Expectation [Iskovskih 1980’s]: Rational cubic fourfolds $Y/\mathbb{C}$ have $\dim H^4(Y)_{alg} \geq 2$; thus a very general cubic is expected to be irrational.

- Rational cubic fourfolds $Y/\mathbb{C}$ are expected to have an associated $K_3$ surface $S/\mathbb{C}$ [Hassett, Kuznetsov, Addington-Thomas]

- Hodge theory: $H_2(S, \mathbb{Z})_{prim} \subset H^4(Y, \mathbb{Z})$ [Hassett] (this condition implies $\dim H^4(Y)_{alg} \geq 2$)

- Derived categories: $D^b(S) \subset D^b(X)$ [Kuznetsov]

- Known: in all examples of rational cubic fourfolds there is an associated $K_3$ surface $E$.
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Associated $K3$ surface

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(Ir)-rationality of cubic fourfolds, continued

Theorem

1. Over a field $k$ satisfying the Cancellation Conjecture, rational smooth cubic fourfolds $Y$ have $F(Y)$ birational to a Hilbert scheme $Hilb_2(S)$ of two points on a $K3$ surface $S$.

2. If $k = \mathbb{C}$ and $\mathbb{C}$ satisfies the Cancellation Conjecture, then $S$ is associated to $Y$ in the sense of Hodge Theory. In particular: very general cubic fourfold is irrational.
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Proof.

- $Y$ rational, $k$ satisfies the Cancellation Conjecture $\implies F(Y)$ stably birational to $S \times S'$ or $\text{Hilb}_2(S) \implies$ they are birational
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- [Hassett] $\implies$ such a $K3$ surface exists only for countable union of divisors in the moduli space
Questions

- Should we expect a very general cubic 5-fold, 6-fold, 7-fold etc to be irrational? Are all smooth odd-dimensional cubics irrational?

- Derived category of coherent sheaves on $F(Y)$

- Chow groups of $F(Y)$
Example: smooth cubics with two disjoint planes

- Let $Y$ be a smooth cubic fourfold containing two disjoint planes $P_1, P_2$.
- Such cubics are rational and they form a codimension 2 subset in the moduli.
- The “natural” $K3$ surface attached to $Y$ is the surface of secants to $P_1 \cup P_2$:

$$S = \{ L \subset Y : L \cap P_1 \neq \emptyset, L \cap P_2 \neq \emptyset \} \subset F(Y)$$

$F(Y)$ is not isomorphic to any $\text{Hilb}^2(K3)$ [Hassett]

However, $F(Y)$ is birational to $\text{Hilb}^2(S)$!

Proof: reduce to the case of a smooth cubic surface marked by two lines $E_1, E_2$. A pair of lines $L_1, L_2$ intersecting $E_1, E_2$ determines a unique line $L$ which does not intersect all four $E_1, E_2, L_1, L_2$. 

E.Shinder (Edinburgh)  
Fano variety of lines and rationality problem for a cubic
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THE END