Fano variety of lines and rationality problem for a cubic hypersurface

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What we know and what we'd like to know about cubic hypersurfaces

The Grothendieck ring of varieties and realizations

The Y-$F(Y)$ relation

Applications to rationality of a cubic

Cubic hypersurfaces

- k: arbitrary field
- $Y \subset \mathbb{P}^{d+1}$: smooth cubic hypersurface of dimension d

Question

Are smooth cubics Y rational varieties (i.e. $k(Y) \simeq k(t_1, \dots k_d)$) ?

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Remarks

- The answer is not known for $d \geq 4$.
- Singular cubics are always rational: projecting from a singular point gives a birational map $Y \dashrightarrow \mathbb{P}^d$.
- If $d \geq 2$, and k is algebraically closed, then smooth cubics Y are unirational varieties: i.e. $k(Y)$ is a finite index subfield of $k(t_1, \dots k_d)$.

Rationality of d-dimensional cubics

- $d = 1, 2$: trivial (elliptic curve irrational, cubic surface rational over $\bar{k}$)
- $d = 3$: cubic threefold - irrational. This is a very subtle result by Clemens-Griffiths (char. 0) and Murre (char. p)
- $d = 4$: cubic fourfold - unknown. There are divisors in the moduli space parametrizing rational ones [Tregub, Beauville-Donagi, Zarhin, Hassett]. Very general cubic fourfold: irrational ??

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Constructions of rational cubics in any **even** dimension $d = 2r$

- Cubics $Y \subset \mathbb{P}^{2r+1}$ containing two disjoint r-planes P_1, P_2: birational morphism $P_1 \times P_2 \to Y$ maps $(a, b) \in P_1 \times P_2$ to the third intersection point of $\langle a, b \rangle$ with Y.
- Cubics $Y \subset \mathbb{P}^{2r+1}$ containing r-dimensional rational scrolls with one apparent double point $W \subset \mathbb{P}^r$: there is a birational morphism $\mathrm{Sym}^2(W) \to Y$
- Such scrolls exist for any r [Severi, Edge, Russo]

Fano variety of lines

Definition

Associated to a cubic hypersurface $Y \subset \mathbb{P}^{d+1}$ there is its Fano variety of lines:

$$F(Y) := \{L \in Gr(1, \mathbb{P}^{d+1}) : L \subset Y\} \subset Gr(1, \mathbb{P}^{d+1})$$

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Properties of $F(Y)$ [Fano, Altman-Kleiman, Barth-Van de Ven]

- $F(Y) \neq \emptyset$ for $d \geq 2$
- $F(Y)$ is connected for $d \geq 3$
- Y smooth $\implies$ $F(Y)$ smooth projective of dimension $2(d-2)$

Fano varieties of lines on a d-dimensional cubic Y

- $d = 1, 2$: no lines on an elliptic curve; 27 lines on a cubic surface over $\bar{k}$.
- $d = 3$: cubic threefold - $F(Y)$ is a surface of general type with $H^1(F(Y), \mathbb{Z}) \simeq H^3(Y, \mathbb{Z})$ [Fano, Clemens-Griffiths]
- $d = 4$: cubic fourfold - $F(Y)$ is an irreducible holomorphic symplectic 4-fold of $K3$ type and $H^2(F(Y), \mathbb{Q}) \simeq H^4(Y, \mathbb{Q})$ [Beauville-Donagi]
- $d \geq 5$: $F(Y)$: $2(d-2)$-dimensional variety with $-K_{F(Y)} > 0$.

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The Fano variety appears naturally in all geometric studies of cubic hypersurfaces.

Question

What is the relation between the geometry of $F(Y)$ and the geometry of Y?

The Y-$F(Y)$ relation

joint work with S.Galkin (HSE, Moscow)

- Find relations between $F(Y)$ and Y in the Grothendieck ring of varieties $K(Var/k)$ and in $K(Var/k)[\mathbb{L}^{-1}]$
- Use these relations to compute invariants of $F(Y)$, e.g. the Hodge structure
- Deduce criteria for rationality of Y modulo a general motivic conjecture

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The Grothendieck ring $K(Var/k)$

Let k be an arbitary field. $K(Var/k)$ is:

- Generators: $[X]$, X/k quasi-projective variety
- Relations: $[X] = [Z] + [U]$ for any closed $Z \subset X$ with open complement U
- Product: $[X] \cdot [Y] = [X \times Y]$

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- $\mathbb{L} := [\mathbb{A}^1]$ - the Lefschetz class (sometimes called the Tate class)
- $[\mathbb{P}^n] = \sum_{k=0}^{n} [\mathbb{A}^k] = \sum_{k=0}^{n} \mathbb{L}^k$
- We don't know: is $\mathbb{L} \in K(Var/k)$ a zero-divisor? Still, often one inverts $\mathbb{L}$ and considers $K(Var/k)[\mathbb{L}^{-1}]$ (motivic integration, motivic Hall algebras).

Properties of the Grothendieck ring

- If $X \to B$ a Zarisky locally trivial fibration with fiber F, then

$$[X] = [F] \cdot [B]$$

- If $Y = Bl_Z(X)$ is a blow up of a smooth subvariety Z in a smooth variety X, then

$$[Y] - [\mathbb{P}(N_{Z/X})] = [X] - [Z]$$

Symmetric powers

- For X/k and $n \geq 0$ we consider the space $Sym^n(X) = X^n/\Sigma_n$.
- By definition $Sym^n(X)$ is the space parametrizing unordered n-tuples of points on X.
- $Sym^n(X)$ are singular if $dim(X) \geq 2$; partial desingularization: $Hilb_n(X) \to Sym^n(X)$ ($Hilb_n(X)$ parametrizes length n subschemes of X)

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Symmetric operations on $K(Var/k)$

The operations $X \mapsto Sym^n(X)$ descend to $K(Var/k)$ and satisfy:

$$Sym^n(\alpha + \beta) = \sum_{i+j=n} Sym^i\alpha \cdot Sym^j\beta; \;\; \alpha, \beta \in K(Var/k)$$

Realizations $K(Var/k) \to R$

A realization is a ring homomorphism $\mu : K(Var/k) \to R$.

Equivalently, a realization is a function $\mu : Var/k \to R$ satisfying

1. $\mu(X) = \mu(Z) + \mu(U)$ for each closed $Z \subset X$ with complement U
2. $\mu(X \times Y) = \mu(X) \cdot \mu(Y)$

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Examples of realizations

- Counting points over a finite field $k = \mathbb{F}_q$: $R = \mathbb{Z}$, $\# : X \mapsto \#X(\mathbb{F}_q)$.
- Topological Euler characteristic for $k = \mathbb{R}$ or $k = \mathbb{C}$: $R = \mathbb{Z}$, $\chi : X \mapsto \chi_c(X(k))$.
- Hodge realization for $k = \mathbb{C}$: $R = K(HS/\mathbb{Q})$ (the Grothendieck ring of rational pure polarizable Hodge structures), $\mu_{Hdg} : X \mapsto gr_W H^*(X, \mathbb{Q})$.

The Grothendieck ring and birational geometry

Unfortunately, we don't know:

- Does $[X] = [Y]$ imply that X and Y admit stratifications $\{X_i\}$, $\{Y_j\}$ with $X_i \simeq Y_i$ after renumbering?
- Even weaker: Does $[X] = [Y]$ imply that X and Y are birational?
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Some good news (the theorem of Larsen-Lunts)

- $[X] = [Y] \implies X$, Y stably birational.

  This means: $X \times \mathbb{P}^k$ is birational to $Y \times \mathbb{P}^l$ for some k, l. For varieties of non-negative Kodaira dimension, this is the same as birational.
- More precisely: let k be a field of characteristic zero. If X and $Y_1, \ldots Y_m$ are smooth projective irreducible varieties and $[X] \equiv \sum_{j=1}^{m} n_j [Y_j]\ (mod\, \mathbb{L})$, for some $n_j \in \mathbb{Z}$, then X is stably birational to one of the Y_j.

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Expressing $Sym^2(Y)$ in terms of $F(Y)$, Y in $K(Var/k)$

Theorem

Let Y/k be a (possibly singular) cubic hypersurface of dimension d. We have the following relations in $K(Var/k)$:

1. $[Hilb_2(Y)] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F(Y)]$
2. $[Sym^2(Y)] = (1 + \mathbb{L}^d)[Y] + \mathbb{L}^2[F(Y)] - \mathbb{L}^d[Sing(Y)]$.

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Immediate consequences

- Number of lines on real and complex cubic surfaces, smooth or singular.
- E.g. a smooth real cubic surface has 3, 7, 15 or 27 real lines (known since [Schläffli 1863]!)
- Euler characteristic of $\chi(F(Y))$, number of points of $F(Y)$ over finite fields, number of components of Y over $\mathbb{R}$ etc.

Idea of the proof

- We prove $[Hilb_2(Y)] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F(Y)]$, the version with $Sym^2(Y)$ follows easily.

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- In other words, we constructed a birational isomorphism

$$\phi : Hilb_2(Y) \dashrightarrow Fl(Y) = \{(y \in Y \cap L,\ L \subset \mathbb{P}^{d+1})\}$$

$Fl(Y) \to Y$ is a $\mathbb{P}^d$-bundle, hence $[Hilb_2(Y)] = [\mathbb{P}^d][Y] +$ extra terms.

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- The extra terms come from the indeterminacy loci of ϕ and ϕ^{-1}. Both are bundles over $F(Y)$. This gives the desired relation.

The class $\mathcal{M}_Y$ of a d-dimensional cubic Y $(d \geq 2)$

- Our next goal is to express $[F(Y)]$ in terms of $[Y]$. This is only possible in $K(Var/k)[\mathbb{L}^{-1}]$.
- We introduce an auxiliary class $\mathcal{M}_Y := \frac{[Y]-[\mathbb{P}^d]}{\mathbb{L}} \in K_0(Var/k)[\mathbb{L}^{-1}]$:

$$[Y] = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_Y \in K_0(Var/k)[\mathbb{L}^{-1}].$$

  $\mathcal{M}_Y$ carries the information about the interesting part of $[Y]$.

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- The intuition comes from Hodge theory. Weak Lefschetz:

$$H^*(Y, \mathbb{Q}) = H^*(\mathbb{P}^d, \mathbb{Q}) \oplus H^d(Y, \mathbb{Q})^{prim}.$$

- We have $\mathcal{H}_Y = \mu_{Hdg}(\mathcal{M}_Y)$, where $\mathcal{H}_Y = H^d(Y, \mathbb{Q})^{prim}(1)$.

The Y-$F(Y)$ relation revisited

Expressing $F(Y)$ in terms of $\mathcal{M}_Y$ in $K(Var/k)[\mathbb{L}^{-1}]$

Theorem

Let $\mathcal{M}_Y = \frac{[Y]-[\mathbb{P}^d]}{\mathbb{L}} \in K_0(Var/k)[\mathbb{L}^{-1}]$*. Then:*

$$[F(Y)] = Sym^2(\mathcal{M}_Y + [\mathbb{P}^{d-2}]) - \mathbb{L}^{d-2} \in K_0(Var/k)[\mathbb{L}^{-1}].$$

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Corollary

If Y *is a smooth cubic* d*-fold over* $\mathbb{C}$*. Let* $\mathcal{H}_Y = H^d(Y,\mathbb{Q})^{prim}(1)$*. Then:*

$$H^*(F(Y),\mathbb{Q}) \simeq Sym^2(\mathcal{H}_Y) \oplus \bigoplus_{k=0}^{d-2} \mathcal{H}_Y(-k) \oplus \bigoplus_{k=0}^{2d-4} \mathbb{Q}(-k)^{a_k}$$

where $a_k \in \mathbb{Z}$ *are given by an explicit formula.*

In particular: $\mathcal{H}_Y \subset H^{d-2}(F(Y),\mathbb{Q})$ *and* $Sym^2(\mathcal{H}_Y) \subset H^{2(d-2)}(F(Y),\mathbb{Q})$*.*

Hodge structure of $F(Y)$: $Y/\mathbb{C}$ smooth cubic threefold

- $H^3(Y, \mathbb{Q})$: weight 3 and Hodge numbers $(0, 5, 5, 0)$
- $\mathcal{H}_Y$: weight 1 and Hodge numbers $(5, 5)$
- $F(Y)$ smooth projective surface of general type with

$$H^*(F(Y), \mathbb{Q}) = \left[\begin{array}{c|ccccc|c} H^4 & & & 1 & & & \mathbb{Q}(-2) \\ H^3 & & 5 & & 5 & & \mathcal{H}_Y(-1) \\ H^2 & 10 & & 25 & & 10 & Sym^2(\mathcal{H}_Y) \\ H^1 & & 5 & & 5 & & \mathcal{H}_Y \\ H^0 & & & 1 & & & \mathbb{Q} \end{array} \right]$$

- This was known before [Clemens-Griffiths]

Hodge structure of $F(Y)$: $Y/\mathbb{C}$ smooth cubic fourfold

- $H^4(Y,\mathbb{Q})$: weight 4 and Hodge numbers $(0,1,21,1,0)$,
- $\mathcal{H}_Y$ has weight 2 and Hodge numbers $(1,20,1)$
- $F(Y)$ irreducible holomorphic symplectic fourfold with $H^*(F(Y),\mathbb{Q})$:

$$\left[\begin{array}{c|ccccc|c} H^8 & & & 1 & & & \mathbb{Q}(-4) \\ H^6 & & 1 & 21 & 1 & & \mathcal{H}_Y(-2)\oplus\mathbb{Q}(-3) \\ H^4 & 1 & 21 & 232 & 21 & 1 & Sym^2(\mathcal{H}_Y)\oplus\mathcal{H}_Y(-1)\oplus\mathbb{Q}(-2) \\ H^2 & & 1 & 21 & 1 & & \mathcal{H}_Y\oplus\mathbb{Q}(-1) \\ H^0 & & & 1 & & & \mathbb{Q} \end{array}\right]$$

  and $H^{2p+1}(F(Y),\mathbb{Q})=0$.
- Beauville-Donagi deduced this from the fact that $F(Y)$ is deformation equivalent to $Hilb_2(K3)$.

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k: a field of characteristic zero.

Theorem (Weak Factorization Theorem – Wlodarczyk et al)

If X and Y are smooth projective birational varieties, then Y can be obtained from X using a finite number of blow ups and blow downs with smooth centers.

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Corollary

If X is a rational smooth projective d-dimensional variety, then

$$[X] = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_X$$

where $\mathcal{M}_X$ is a linear combination of classes of smooth projective varieties of dimension $d - 2$.

Criterion for irrationality of a cubic in any dimension

The Cancellation Conjecture

- $\mathbb{L}$ is not a zero-divisor in $K(Var/k)$ (at the moment this is not known for any field).

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Theorem

Let Y be a smooth rational d-dimensional cubic ($d \geq 3$) over a field satisfying the Cancellation Conjecture.

Then $F(Y)$ is stably birationally equivalent to one of the

$$Hilb_2(V) \text{ or } V \times V'$$

where V, V' are smooth projective $(d-2)$-dimensional varieties.

Steps of the proof

- Weak Factorization: Y rational $\implies Y = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_Y$ for

$$\mathcal{M}_Y = \sum_{i=1}^{n} [V_i] - \sum_{j=1}^{m} [W_j], \ \ \dim V_i = \dim W_j = d - 2.$$

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- Recall the Y-$F(Y)$ relation:

$$[F(Y)] = Sym^2(\mathcal{M}_Y + [\mathbb{P}^{d-2}]) - \mathbb{L}^{d-2} \in K(Var/k)[\mathbb{L}^{-1}]$$

If $\mathbb{L}$ is not a zero-divisor, this in fact holds in $K(Var/k)$!

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- Reduce mod $\mathbb{L}$ and compute:

$$\begin{aligned}[F(Y)] &\equiv Sym^2(\mathcal{M}_Y + \mathbb{P}^{d-2}) \ (mod\, \mathbb{L}) \\ &\in \big\langle [V \times V'], [Hilb_2 V] \big\rangle \ (mod\, \mathbb{L})\end{aligned}$$

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- [Larsen-Lunts] $\implies$ $F(Y)$ is stably birational to one of the $V \times V'$ or $Hilb_2 V$.

Irrationality of cubic threefolds

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- Hodge numbers $\implies$ the only possibility is $Sym^2(C)$, $g(C) = 5$
- Impossible! $Sym^2(C)$ is a minimal model, but

$$25 = h^{1,1}(F(Y)) < h^{1,1}(Sym^2(C)) = 26.$$

□

(Ir)-rationality of cubic fourfolds

- In the moduli of all cubic fourfolds there are countably many divisors which parametrize rational ones [Tregub, Zarhin, Hassett]
- Expectation [Iskovskih 1980's]: Rational cubic fourfolds $Y/\mathbb{C}$ have $\dim\, H^4(Y)_{alg} \geq 2$; thus a very general cubic is expected to be irrational

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Associated $K3$ surface

- Rational cubic fourfolds $Y/\mathbb{C}$ are expected to have an associated $K3$ surface $S/\mathbb{C}$ [Hassett, Kuznetsov, Addington-Thomas]
- Hodge theory: $H^2(S,\mathbb{Z})^{prim}(-1) \subset H^4(Y,\mathbb{Z})$ [Hassett] (this condition implies $\dim\, H^4(Y)_{alg} \geq 2$)
- Derived categories: $D^b(S) \subset D^b(X)$ [Kuznetsov]
- Known: in all examples of rational cubic fourfolds there is an associated $K3$

(Ir)-rationality of cubic fourfolds, continued

Theorem

1. *Over a field k satisfying the Cancellation Conjecture, rational smooth cubic fourfolds Y have $F(Y)$ birational to a Hilbert scheme $Hilb_2(S)$ of two points on a $K3$ surface S.*
2. *If $k = \mathbb{C}$ and $\mathbb{C}$ satisfies the Cancellation Conjecture, then S is associated to Y in the sense of Hogde Theory. In particular: very general cubic fourfold is irrational.*

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Proof.

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- S is associated to Y in the sense of Hodge Theory
- [Hassett] $\implies$ such a $K3$ surface exists only for countable union of divisors in the moduli space

□

Questions

- Should we expect a very general cubic 5-fold, 6-fold, 7-fold etc to be irrational ? Are all smooth odd-dimensional cubics irrational?
- Derived category of coherent sheaves on $F(Y)$
- Chow groups of $F(Y)$

Example: smooth cubics with two disjoint planes

- Let Y be a smooth cubic fourfold containing two disjoint planes P_1, P_2.
- Such cubics are rational and they form a codimension 2 subset in the moduli
- The "natural" $K3$ surface attached to Y is the surface of secants to $P_1 \cup P_2$:

$$S = \{L \subset Y : L \cap P_1 \neq \emptyset, L \cap P_2 \neq \emptyset\} \subset F(Y)$$

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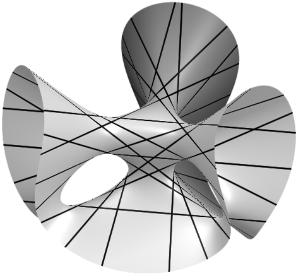
- $F(Y)$ is **not** isomorphic to **any** $Hilb_2(K3)$ [Hassett]
- However, $F(Y)$ is birational to $Hilb_2(S)$!
- Proof: reduce to the case of a smooth cubic surface marked by two lines E_1,E_2. A pair of lines L_1, L_2 intersecting E_1,E_2 determines a unique line L which does not interesect all four E_1, E_2, L_1, L_2.



THE END