DERIVED CATEGORIES AND ALGEBRAIC K-THEORY OF SINGULARITIES EPSRC PROPOSAL

EVGENY SHINDER

1. Background

The study of derived categories of coherent sheaves on algebraic varieties goes back to the work of Mukai, Bondal–Orlov, Bridgeland–Maciocia. Spectacular results have been obtained for derived categories of nonsingular varieties such as reconstruction of the variety from its derived category in the Fano or anti-Fano case by Bondal-Orlov, derived equivalence of 3-fold flops by Bridgeland, derived categories as a tool controlling Birational Geometry by Kawamata, Homological Projective Duality as a general machinery to obtain derived equivalence by Kuznetsov, applications of Bridgeland stability conditions in Birational Geometry of moduli spaces by Bayer–Macri, relation via tilting to noncommutative algebra by van den Bergh and Iyama–Wemyss. Potential relation to rationality problems, specifically for cubic fourfolds has been suggested by Kuznetsov and derived categories prediction was shown to match up Hodge theory by the work of Addington and Thomas.

To a large extent the main emphasis of the most existing work is on studying derived categories of nonsingular algebraic varieties. Yet, singular varieties are ubiquitous in Algebraic Geometry, in particular in the Minimal Model Program (MMP). The contractions in MMP correspond well with semiorthogonal decompositions and lack of understanding of derived catgories of singular varieties blocks the progress in this area.

Quite recently derived categories of singular varieties and their semiorthogonal decompositions came into focus in work of Kawamata [9], Kuznetsov [13] and my joint work with Karmazyn and Kuznetsov [14]. The question of existence of interesting semiorthogonal decompositions has nontrivial relations to K-theoretic invariants of singularites such as the negative K_{-1} group and the Brauer group. In the simplest instances this translates into questions about properties of singularities such as factoriality, and the relative positions of the singular points.

The language and techniques of algebraic K-theory predate developments in derived categories, and for a long time they were developing in parallel. Modern algebraic K-theory goes back to the work of Quillen who constructs higher K-theory groups of an exact category. The difference between K-theory of coherent sheaves (called G-theory) and that of perfect complexes or vector bundles (called K-theory) for singular varieties was always part of the landscape. Nowadays K-theory of singular varieties is a well-developed and active area which has seen very recently spectacular progress in the work of Kerz, Strunk and Tamme on Weibel's conjecture and pro-cdh descent for algebraic K-theory.

The abstract machinery of algebraic K-theory has been developed much further too, via the work of Thomason-Trobaugh and culminating in Schlichting's construction of algebraic K-theory groups, including negative K-theory of a dg-enhanced triangulated category [18]. This machinery has been applied in my joint work with my PhD student Pavic [16] to the Orlov singularity category to get results concerning both derived categories and derived categories of singularities as well as algebraic K-theory of singular varieties.

1.1. **The aim of this proposal.** In this proposal I suggest a detailed study of triangulated categories and algebraic K-theory attached to singular varieties. More precisely, the study has the following five objectives:

(1) Constructing semiorthogonal decompositions for singular projective varieities, specifically for toric singular threefolds.
(2) Exploiting methods from Algebraic K-theory, such as K_{-1} group and the Brauer group to provide obstructions for existence to these decompositions in various situations.
(3) Classifying maximally nonfactorial nodal Fano threefolds and constructing Kawamata decompositions for each of them.
(4) Clarifying relationships between algebraic variety and its resolution, on the level of derived categories and algebraic K-theory, in particular studying the Bondal-Orlov localization conjecture and its homological version.
(5) Exhibiting geometric meaning of the K_{-1} group, in particular in relation to types and positions of singularities, as well as studying behaviour of K_{-1} in families.

1.2. **Methodology.** The main tools and techniques I rely in this study are:

- Derived categories of coherent sheaves $\mathcal{D}^b(X)$, perfect complexes $\mathcal{D}^{\mathrm{perf}}(X)$ and Orlov's triangulated singularity category $\mathcal{D}^{\mathrm{sg}}(X) = \mathcal{D}^b(X)/\mathcal{D}^{\mathrm{perf}}(X)$ [17]
- Thomason-Trobaugh algebraic K-theory and Schlichting's K-theory groups [18], specifically the K_{-1} group which controls idempotent completeness of $\mathcal{D}^{\mathrm{sg}}(X)$
- Algebraic K-theory tools such as Suslin-Voevodsky cdh cohomology and Brauer group in order to compute and understand algebraic K-theory
- Resolution of singularities, including Kunzetsov-Lunts [15] noncommutative resolutions, the relative singularity category of Burban and Kalck and its algebraic K-theory
- Threefolds with isolated singularities, such as compound du Val, Toric varieties and nodal Fano threefolds as sources of examples
- Kuznetsov's Homological Projective Duality [12] to construct semiorthogonal decompositions of singular varieties

2. Detailed description of the project

One of the main objects of study for the project is the notion of a *Kawamata semiorthogonal decomposition* inspired by the recent work of Kawamata [10].

Let X be a Gorenstein projective variety over an algebraically closed field of characteristic zero. One good example of such a variety is the one with isolated ordinary double points, that is points with an analytic local equation

$$x_1^2 + \cdots + x_{n+1}^2 = 0$$

in the affine space (here $n = \dim(X)$). Ordinary double points are the most common type of singularities. Varieties with ordinardy double points are also called *nodal*. Most of what is said below will be illustrated in the nodal case.

We write $\mathcal{D}^b(X)$ for the derived category of coherent sheaves and $\mathcal{D}^{\mathrm{perf}}(X)$ for its full subcategory consisting of perfect complexes.

We say that X has a Kawamata semiorthogonal decomposition if there is an admissible semiorthogonal decomposition

$$\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_r \rangle \tag{2.1}$$

where $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$, and each $\mathcal{B}_i$ is equivalent to $\mathcal{D}^b(R_i)$ for some finite dimensional possibly noncommutative algebra R_i. This definition is not yet in the literature and will appear in forthcoming joint work with Kalck and Pavic. Kawamata decomposition may be thought of splitting derived category in its "nonsingular part" $\mathcal{A}$ and algebras R_i which carry information about singularities of X. One other way to think about the definition is that in the special case $\mathcal{A} = 0$, (2.1) is a generalization of a concept of a full exceptional collection; the full exceptional collection case corresponds to all $R_i = k$.

The current state of the art is the following. In dimension one, Burban studied derived categories of singular curves [4]. In the language of this proposal, Burban's main result has been that nodal chains of $\mathbb{P}^1$'s admit Kawamata semiorthogonal decompositions. No other singular curves are known to admit one.

It follows from the main result of the PI, Karmazyn and Kuznetsov [14] that for a toric projective surface X a Kawamata decomposition exists if and only if the Brauer group $\mathrm{Br}(X)$ is zero (the Gorenstein condition is not needed here). For example, any weighted projective plane $\mathbb{P}(a, b, c)$ admits a Kawamata decomposition into three finite-dimensional noncommutative algebras R_1, R_2, R_3 which depend on a, b, c.

In dimension three, Kawamata has given two examples of Fano threefolds, both toric with a single node and having a Kawamata decomposition [10]. These are the nodal 3-dimensional quadric and the blow up of $\mathbb{P}^3$ in two points followed by a blow down of the line. It was important in the constructions that considered varieties are nonfactorial, that is there exist Weil divisors which are not Cartier. However, this link was not explained conceptually by Kawamata. It will be explained below using algebraic K-theory.

Objective 1. Kawamata decompositions: existence. I propose to study Kawamata decompositions in more general setup. One important setting is the case of nodal toric threefolds, generalizing the work of PI, Karmazyn and Kuznetsov [14] as well as Kawamata's [10]. Here the strategy would be similar to that of the PI, Karmazyn and Kuznetsov, where one constructs a semiorthogonal decomposition for the resolution and then makes a Verdier quotient to construct semiorthogonal decomposition of the singular variety. The fact that the derived category of singular varieities with rational singularities is a Verdier quotient of that of the resolution is known as the Bondal-Orlov localization conjecture, see Conjecture 2.2 below. It is known in some cases, including threefolds admitting a small resolution. Given that everything about toric varieties is explicit and combinatorial it is quite likely that existence of a Kawamata decomposition would be a condition on the fan of the corresponding toric threefold.

The next natural and important thing to study is Kawamata decompositions as limits of full exceptional collection of the nearby smooth fibers. Indeed, in the known cases it seems that the algebras $\mathcal{D}^b(R_i)$ coming from the definition of Kawamata decomposition (2.1) are obtained as several exceptional objects get glued together. The classical counterpart of this, the theory of vanishing cycles in

Hodge structures may be the right intuition to approach these questions. Formally one should define Kawamata decomposition over the base and study generalizations and specializations results for these.

Objective 2. Kawamata decomposition: obstructions. In the forthcoming work of PI, Kalck and Pavic we prove that if a nodal threefold admits a Kawamata decomposition, then X is what we call *maximally nonfactorial*. In the nodal case, this means that the map

$$\mathrm{Cl}(X) \to \mathbb{Z}^{\mathrm{Sing}(X)}$$

from the class group to the direct sum of local class groups at all the nodes is surjective. This explains why both Kawamata's examples are nonfactorial.

Our proof relies on vanishing of the K_{-1} group of X of any variety X admitting a Kawamata decomposition; this in turn comes from analyzing Orlov's singularity category $\mathcal{D}^{\mathrm{sg}}(X)$.

A similar study must be undertaken in other dimensions and for other types of singularities. Here one typical phenomenon is that of Knörrer periodicity: ordinary double points of dimension n behave differently depending on $n \pmod 2$. Thus controlling curves and/or threefolds (resp. surfaces) allows to prove nontrivial results about all odd dimensional (resp. even dimensional) ordinary double points The same phenomenon exists for other types of singularities, most notably for compound du Val singularities.

Objective 3. Classification of maximally nonfactorial nodal Fano threefolds. It is clear that studying semiorthogonal decomposition for singular varieties is quite important, but how common are Kawamata decompositions? For example, what about singular Fano threefolds? Here is one explicit thing one can do.

It follows from the link between Kawamata decompositions to class groups, explained above that nodal threefolds that have a Kawamata decompositions are maximally nonfactorial. In practical terms this implies that the following holds

$$\operatorname{rk}\mathrm{Cl}(X) = \operatorname{rk}\mathrm{Pic}(X) + \#\mathrm{Sing}(X). \tag{2.2}$$

In fact LHS is always bounded by the RHS, and in the classical language of birational geometry of 3-folds, maximally nonfactorial threefolds have maximal *defect*, that is their defect, defined as $\operatorname{rk}\mathrm{Cl}(X) - \operatorname{rk}\mathrm{Pic}(X)$ is equal to the number of singular points.

Known results about defect for threefold hypersurfaces and double covers imply that (2.2) never holds for such threefolds. Looking at some other examples Ivan Cheltsov suggested the following.

Conjecture 2.1. *Maximally nonfactorial nodal Fano threefolds of Picard rank one are precisely the nodal quadric Q, nodal V_5 and nodal V_{22}. Furthermore, these varieties all admit a Kawamata decomposition.*

We note that the nonsingular Fano varieties of these types are precisely three-dimensional *minifolds* [7]: together with $\mathbb{P}^3$ they are the only threefolds that admit a full exceptional collection of the minimal length 4. It would be quite exciting to understand this interplay between Kawamata decompositions and Fano minifolds. This relation quite likely should go via deformation and specialization for Kawamata decompositions and using Kuznetsov's Homological Projective Duality applied to singular varieties, in particular to the nodal V_5.

Objective 4. Bondal-Orlov localization conjecture. The main method for constructing semiorthogonal decompositions of singular varieties is to descend decompositions from the resolution. From this perspective the following conjecture is the key.

Conjecture 2.2 (Bondal-Orlov localization)**.** *If X is a variety with rational singularities and $\pi : Y \to X$ is a resolution of singularities, then $R\pi_* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ is a Verdier quotient by* $\mathrm{Ker}(R\pi_*)$.

The importance of the Bondal-Orlov localization however goes well-beyond constructing semiorthogonal decompositions for singular varieties, and is crucial for interpreting the Minimal Model Program (MMP) categorically. The conjecture also has to do with existence of a t-structure on $\mathrm{Ker}(R\pi_*)$ which has been important in Bridgeland's proof of derived equivalence for 3-fold flops [5]. It is not clear what is the expectation about existence of such t-structure in general, and this question definitely should be investigated in relation to the Bondal-Orlov localization.

The conjecture first appeared in the Bondal-Orlov 2002 ICM talk [3]. It has been studied by Efimov who proved it for cones over smooth Fano varieites [6]. Very recently Kawamata has proved the weaker version of essential surjectivity of $R\pi_*$, namely that it is surjective up to summands [10]. In joint work with my PhD student Pavic [16] we proved Conjecture 2.2 for quotient singularities (not necessarily isolated). Together with Pavic we plan to study the toric case of the conjecture.

One important approach to this conjecture which has been employed by Efimov is to use Kuznetsov-Lunts noncommutative resolutions [15] and do induction on thickenings of the singular locus; I find it remarkable that the same induction plays the essential role in pro-cdh descent for Algebraic K-theory [11]. Such an induction on thickenings could be used in dealing with the toric case too.

A weaker version is the consequence of Conjecture 2.2 seen in algebraic K-theory:

Conjecture 2.3 (Homological Bondal-Orlov localization)**.** *In the same setting, there is an exact sequence*

$$\cdots \to \mathrm{K}_i(\mathrm{Ker}(R\pi_*)) \to \mathrm{K}_i(Y) \to \mathrm{G}_i(X) \to \mathrm{K}_{i-1}(\mathrm{Ker}(R\pi_*)) \to \cdots \to \mathrm{K}_0(Y) \to \mathrm{G}_0(X) \to 0.$$

Here the algebraic K-theory notation is used: $\mathrm{K}_i(Y)$ stands for K-theory groups of $\mathcal{D}^{\mathrm{perf}}(Y)$ while $\mathrm{G}_i(X)$ is K-theory of $\mathcal{D}^b(X)$. K-theory groups of the kernel category $\mathrm{Ker}(R\pi_*)$ and the exact sequence as a consequence of a Verdier quotient respecting dg-enhancements is using the work of Schlichting [18].

In fact even the last surjectivity $\mathrm{K}_0(Y) \to \mathrm{G}_0(X)$ is an open question, but the abundance of techniques available here such as comparison with Chow groups makes one quite hopeful. Proving or disproving this Homological version of the conjecture does not seem out of reach. Then one would be able to deal with the original version, Conjecture 2.2.

Objective 5. Geometric meaning and behaviour in families of negative K-theory. One of the key things this study can offer to algebraic K-theory is to exhibit geometric meaning of the negative K-theory groups; these groups can be nonzero only for singular varieties, and according to Weibel's conjecture (now a theorem) $\mathrm{K}_i(X) = 0$ for $i < -\dim(X)$.

The 'homological' meaning of $\mathrm{K}_{-1}(X)$ comes from the work of Thomason and Orlov: $\mathrm{K}_{-1}(X) = 0$ if and only if the Orlov singularity category $\mathcal{D}^{\mathrm{sg}}(X)$ is idempotent complete. However no general geometric criteria for this completeness is known (except in dimensions one and two).

I would like to phrase the vanishing of K_{-1}, and possibly clarify the role of other negative K-theory groups, in terms of simpler cohomological invariants such as the class group, the Picard group, the defect, the Brauer group, and possibly Chow groups of algebraic cycles. In fact in dimensions one and two this has been done by Weibel [21]. One important link between K_{-1}, derived categories and Brauer groups is in my joint work with Karmazyn and Kuznetsov [14] where we study Brauer twisted toric surfaces and as a result of this twisting K_{-1} can drop.

A related question is how negative K-theory, in particular vanishing of K_{-1} behaves in families. The intuition we obtain from examples, and using the interplay between K_{-1} and the defect, is the following: when nodes on a threefold are in general position, K_{-1} stays constant (small constant defect), whereas when nodes come into special configurations, K_{-1} drops (defect increases as more Weil divisors such as planes passing through nodes may appear). However no general specialization results like these seem to be known, and setting up a machinery to deal with these questions is an important task.

Finally, the size of $\mathrm{K}_{-1}(X)$ also has to do with the question of existence and the number of small resolutions, e.g. for nodal threefolds. Roughly speaking this is because having small $\mathrm{K}_{-1}(X)$ means having large class group which translates into the possibility of flopping contracted curves independently in the small resolution. This phenomenon, properly investigated could be considered both as application of the K_{-1} group, as well as geometric meaning of it.

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