Geometry of singularities

Evgeny Shinder

02.07.2019

Plan of the talk

Plan of the talk

1. What is Algebraic Geometry?

Plan of the talk

1. What is Algebraic Geometry?
2. How to study singularities?

Plan of the talk

1. What is Algebraic Geometry?
2. How to study singularities?
3. What are singularities good for?

What is Algebraic Geometry?

What is Algebraic Geometry?

In Algebraic Geometry one studies **algebraic varieties**, that is sets given by polynomial equations.

What is Algebraic Geometry?

In Algebraic Geometry one studies **algebraic varieties**, that is sets given by polynomial equations.

Example

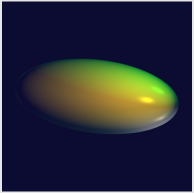
Algebraic surfaces are solutions of $f(x, y, z) = 0$, where f is a polynomial.

What is Algebraic Geometry?

In Algebraic Geometry one studies **algebraic varieties**, that is sets given by polynomial equations.

Example

Algebraic surfaces are solutions of $f(x, y, z) = 0$, where f is a polynomial.



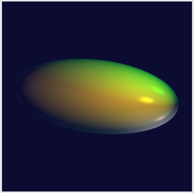
A nonsingular **quadric** surface
$3x^2 + 3y^2 + z^2 - 1 = 0$

What is Algebraic Geometry?

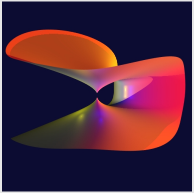
In Algebraic Geometry one studies **algebraic varieties**, that is sets given by polynomial equations.

Example

Algebraic surfaces are solutions of $f(x, y, z) = 0$, where f is a polynomial.



A nonsingular **quadric** surface
$3x^2 + 3y^2 + z^2 - 1 = 0$



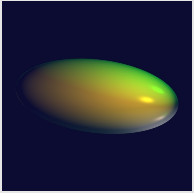
A singular **quartic** surface
$x^2 - x^3 + y^2 + y^4 + z^3 - 10z^4 = 0$

What is Algebraic Geometry?

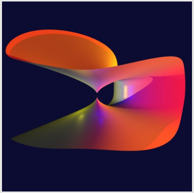
In Algebraic Geometry one studies **algebraic varieties**, that is sets given by polynomial equations.

Example

Algebraic surfaces are solutions of $f(x, y, z) = 0$, where f is a polynomial.



A nonsingular **quadric** surface
$3x^2 + 3y^2 + z^2 - 1 = 0$



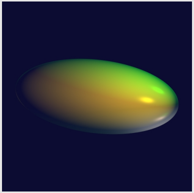
A singular **quartic** surface
$x^2 - x^3 + y^2 + y^4 + z^3 - 10z^4 = 0$

What is Algebraic Geometry?

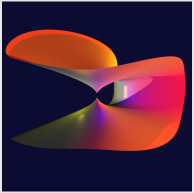
In Algebraic Geometry one studies **algebraic varieties**, that is sets given by polynomial equations.

Example

Algebraic surfaces are solutions of $f(x, y, z) = 0$, where f is a polynomial.



A nonsingular **quadric** surface
$3x^2 + 3y^2 + z^2 - 1 = 0$



A singular **quartic** surface
$x^2 - x^3 + y^2 + y^4 + z^3 - 10z^4 = 0$

In this talk I present some classical narratives about **singularities** in Algebraic Geometry.

Singularities: geometry meets topology and analysis

Singularities: geometry meets topology and analysis

Definition

Singularities: geometry meets topology and analysis

Definition

A polynomial equation $f(x_1, \dots, x_n) = 0$ has a singularity at $p \in \mathbb{R}^n$ if $f(p) = 0$ and all partial derivatives $\frac{\partial f}{\partial x_i}(p) = 0$.

Singularities: geometry meets topology and analysis

Definition

A polynomial equation $f(x_1, \ldots, x_n) = 0$ has a singularity at $p \in \mathbb{R}^n$ if $f(p) = 0$ and all partial derivatives $\frac{\partial f}{\partial x_i}(p) = 0$.

Examples

- The curve $x^2 - y^2 = 1$ is non-singular
- The curve $x^2 - y^2 = 0$ is singular at $(0, 0)$
- The surface $xy - z^2 - z^3 = 0$ is singular at $(0, 0, 0)$

Singularities: geometry meets topology and analysis

Definition

A polynomial equation $f(x_1, \ldots, x_n) = 0$ has a singularity at $p \in \mathbb{R}^n$ if $f(p) = 0$ and all partial derivatives $\frac{\partial f}{\partial x_i}(p) = 0$.

Examples

- The curve $x^2 - y^2 = 1$ is non-singular
- The curve $x^2 - y^2 = 0$ is singular at $(0,0)$
- The surface $xy - z^2 - z^3 = 0$ is singular at $(0,0,0)$

Singularities above are **ordinary double points**: the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ of second derivatives is invertible, and hence the singularity is purely quadratic, after a coordinate change (Taylor series + completing the square!).

Complex and real singularities

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Different types of real ordinary double points:

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Different types of real ordinary double points:

- $x^2 + y^2 = 0$ "minimum"
- $x^2 - y^2 = 0$ "saddle"
- $-x^2 - y^2 = 0$ "maximum"

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Different types of real ordinary double points:

- $x^2 + y^2 = 0$ "minimum"
- $x^2 - y^2 = 0$ "saddle"
- $-x^2 - y^2 = 0$ "maximum"

Three equations $x^2 + y^2$, $x^2 - y^2$, $-x^2 - y^2$ are all equivalent under complex change of variables.

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Different types of real ordinary double points:

- $x^2 + y^2 = 0$ "minimum"
- $x^2 - y^2 = 0$ "saddle"
- $-x^2 - y^2 = 0$ "maximum"

Three equations $x^2 + y^2$, $x^2 - y^2$, $-x^2 - y^2$ are all equivalent under complex change of variables.

A typical question is: which *complex* singularities are equivalent, that is can be translated into one another by a change of coordinates?

Some interesting curve singularities

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Different types of real ordinary double points:

- $x^2 + y^2 = 0$ "minimum"
- $x^2 - y^2 = 0$ "saddle"
- $-x^2 - y^2 = 0$ "maximum"

Three equations $x^2 + y^2$, $x^2 - y^2$, $-x^2 - y^2$ are all equivalent under complex change of variables.

A typical question is: which *complex* singularities are equivalent, that is can be translated into one another by a change of coordinates?

Some interesting curve singularities

- Type A_n $(n \geq 1)$: $x^2 + y^{n+1} = 0$

Complex and real singularities

One can do algebraic geometry over $\mathbb{R}$ or over $\mathbb{C}$ (or over any other field); geometry over $\mathbb{C}$ is *simpler*.

Real and complex ordinary double points in two variables

Different types of real ordinary double points:

- $x^2 + y^2 = 0$ "minimum"
- $x^2 - y^2 = 0$ "saddle"
- $-x^2 - y^2 = 0$ "maximum"

Three equations $x^2 + y^2$, $x^2 - y^2$, $-x^2 - y^2$ are all equivalent under complex change of variables.

A typical question is: which *complex* singularities are equivalent, that is can be translated into one another by a change of coordinates?

Some interesting curve singularities

- Type A_n $(n \geq 1)$: $x^2 + y^{n+1} = 0$
- Type D_n $(n \geq 3)$: $y(x^2 + y^{n-2}) = 0$

Some words about my research

Some words about my research

One of the strands of my research is studying singularities in Algebraic Geometry, along the following two perspectives:

Some words about my research

One of the strands of my research is studying singularities in Algebraic Geometry, along the following two perspectives:

1. I study various algebraic invariants attached to singularties. An example of this way of thinking: the **Jacobi ring** of a singularity. What I actually do: study derived categories of singularities and their K-theory groups.

Some words about my research

One of the strands of my research is studying singularities in Algebraic Geometry, along the following two perspectives:

1. I study various algebraic invariants attached to singularties. An example of this way of thinking: the **Jacobi ring** of a singularity. What I actually do: study derived categories of singularities and their K-theory groups.

2. I use degeneration techniques to study nonsingular varieties, degenerating them to singular ones. An example of this way of thinking: computing **genus of a nonsingular curve**. What I actually do: I study **stable birational types via degeneration to very singular varieties**.

Some words about my research

One of the strands of my research is studying singularities in Algebraic Geometry, along the following two perspectives:

1. I study various algebraic invariants attached to singularties. An example of this way of thinking: the **Jacobi ring** of a singularity. What I actually do: study derived categories of singularities and their K-theory groups.

2. I use degeneration techniques to study nonsingular varieties, degenerating them to singular ones. An example of this way of thinking: computing **genus of a nonsingular curve**. What I actually do: I study **stable birational types via degeneration to very singular varieties**.

I will explain these two ways of thinking.

What it means to study singularities?

What it means to study singularities?

We get a little bit technical now...

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \ldots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \ldots, x_n] / \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \Big).$$

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \ldots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \ldots, x_n]/\Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

The **Milnor number** μ_f is the dimension of the vector space $\mathcal{A}_f$.

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \ldots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \ldots, x_n] / \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \Big).$$

The **Milnor number** μ_f is the dimension of the vector space $\mathcal{A}_f$.

Both the A_n curve singularitiy $x^2 + y^{n+1} = 0$ and the D_n curve singularity $y(x^2 + y^{n-2})$ have Milnor number equal to n.

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \ldots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \ldots, x_n] / \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \Big).$$

The **Milnor number** μ_f is the dimension of the vector space $\mathcal{A}_f$.

Both the A_n curve singularitiy $x^2 + y^{n+1} = 0$ and the D_n curve singularity $y(x^2 + y^{n-2})$ have Milnor number equal to n.

So are A_n and D_n singularities equivalent?

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \dots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \dots, x_n] / \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \Big).$$

The **Milnor number** μ_f is the dimension of the vector space $\mathcal{A}_f$.

Both the A_n curve singularitiy $x^2 + y^{n+1} = 0$ and the D_n curve singularity $y(x^2 + y^{n-2})$ have Milnor number equal to n.

So are A_n and D_n singularities equivalent?

- A_3 and D_3 are equivalent

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \ldots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \ldots, x_n]/\Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

The **Milnor number** μ_f is the dimension of the vector space $\mathcal{A}_f$.

Both the A_n curve singularitiy $x^2 + y^{n+1} = 0$ and the D_n curve singularity $y(x^2 + y^{n-2})$ have Milnor number equal to n.

So are A_n and D_n singularities equivalent?

- A_3 and D_3 are equivalent
- A_n and D_n are not equivalent for $n > 3$ as their Jacobi rings are different.

What it means to study singularities?

We get a little bit technical now...

Jacobi ring and Milnor number

Let $f(x_1, \dots, x_n) = 0$ be a singularity. Then its **Jacobi ring** is defined as the quotient ring of the polynomial ring by the ideal generated by all partial derivates of f:

$$\mathcal{A}_f := \mathbb{C}[x_1, \dots, x_n] / \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \Big).$$

The **Milnor number** μ_f is the dimension of the vector space $\mathcal{A}_f$.

Both the A_n curve singularitiy $x^2 + y^{n+1} = 0$ and the D_n curve singularity $y(x^2 + y^{n-2})$ have Milnor number equal to n.

So are A_n and D_n singularities equivalent?

- A_3 and D_3 are equivalent
- A_n and D_n are not equivalent for $n > 3$ as their Jacobi rings are different.

Another viewpoint: intersecting the singular complex curve $f(x, y) = 0$ with a 3-sphere $S^3 \subset \mathbb{C}^2$ one gets a collection of μ_f circles S^1...

Using degeneration techniques

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

It is often useful to degenerate nonsingular varieties to singular ones. We get a bit technical again...

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

It is often useful to degenerate nonsingular varieties to singular ones. We get a bit technical again...

Genus of a nonsingular algebraic curve

Let X given by $f(x, y) = 0$ be a nonsingular complex algebraic curve. Its set of points X can be thought of as a 2-dimensional manifold.

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

It is often useful to degenerate nonsingular varieties to singular ones. We get a bit technical again...

Genus of a nonsingular algebraic curve

Let X given by $f(x, y) = 0$ be a nonsingular complex algebraic curve. Its set of points X can be thought of as a 2-dimensional manifold.

- Let g be the genus of X (number of "holes") and let d be the degree of the equation f

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

It is often useful to degenerate nonsingular varieties to singular ones. We get a bit technical again...

Genus of a nonsingular algebraic curve

Let X given by $f(x, y) = 0$ be a nonsingular complex algebraic curve. Its set of points X can be thought of as a 2-dimensional manifold.

- Let g be the genus of X (number of "holes") and let d be the degree of the equation f
- The **genus-degree formula** is the relation between d and g:

$$g = \frac{(d-1)(d-2)}{2}$$

Using degeneration techniques

Example of a **degeneration**: nonsingular curve $x^2 - y^2 = 1$ degenerates to a singular one $x^2 - y^2 = 0$ in a family $x^2 - y^2 = t$ (t parameter).

It is often useful to degenerate nonsingular varieties to singular ones. We get a bit technical again...

Genus of a nonsingular algebraic curve

Let X given by $f(x, y) = 0$ be a nonsingular complex algebraic curve. Its set of points X can be thought of as a 2-dimensional manifold.

- Let g be the genus of X (number of "holes") and let d be the degree of the equation f
- The **genus-degree formula** is the relation between d and g:

$$g = \frac{(d-1)(d-2)}{2}$$

- Sketch proof using degeneration: degenerate X to a singular union of curves of smaller genus and use induction.

Summary: Algebraic Geometry of Singularities

Summary: Algebraic Geometry of Singularities

1. Singularities naturally belong to the intersection of analysis, topology and algebraic geometry.

Summary: Algebraic Geometry of Singularities

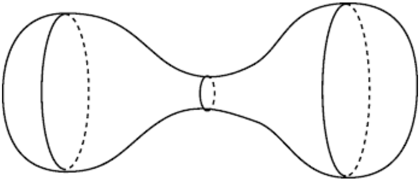
1. Singularities naturally belong to the intersection of analysis, topology and algebraic geometry.
2. Singularities are studied and classified by attaching algebraic invariants to them.

Summary: Algebraic Geometry of Singularities

1. Singularities naturally belong to the intersection of analysis, topology and algebraic geometry.
2. Singularities are studied and classified by attaching algebraic invariants to them.
3. Studying singularities can lead to better understanding of non-singular varieties, via degeneration techniques.

Summary: Algebraic Geometry of Singularities

1. Singularities naturally belong to the intersection of analysis, topology and algebraic geometry.
2. Singularities are studied and classified by attaching algebraic invariants to them.
3. Studying singularities can lead to better understanding of non-singular varieties, via degeneration techniques.



THE END