DERIVED CATEGORIES: LECTURE 1

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References

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1. Derived and triangulated categories

Let $\mathcal{A}$ be an abelian category and let $Kom(\mathcal{A})$ denote the category of complexes of objects in $\mathcal{A}$.

A morphism of complexes $f : A^\bullet \to B^\bullet$ is called a *quasi-isomorphism* if the induced functors $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all $i \in \mathbb{Z}$.

For example, if $P^\bullet \to A$ is a resolution, then the morphism $P^\bullet \to A[0]$ is a quasi-isomorphism.

In homological algebra we identify an object with all its resolutions, this leads naturally to considering all complexes, modulo quasi-isomorphism.

Definition 1.1. The derived category $D(\mathcal{A})$ of $\mathcal{A}$ is obtained by formally inverting all quasi-isomorphisms.

$D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, resp. $D^b(\mathcal{A})$) is defined as a full subcategory of $D(\mathcal{A})$ consisting of objects with cohomology bounded from below (resp. above, resp. above and below).

There is a fully faithful embedding of $\mathcal{A}$ into $D^b(\mathcal{A})$ given by $A \mapsto A[0]$. However, the derived category itself is not at all abelian. Derived category is a so-called *triangulated category* which means that it admits a *shift functor* [1] and given a class of *distinguished triangles*.

The shift functor [1] simply shifts the complex. Beware of the direction of cohomological grading shift: e.g. $A[1]$ is A sitting in degree -1!

Instead of kernels and cokernels we have cones. If $f : A^\bullet \to B^\bullet$ is a morphism of complexes, then the cone of f is defined as:

$$cone(f)^p := B^p \oplus A^{p+1}$$

and the differential maps (b^p, a^{p+1}) to $(d(b^p) + f(a^{p+1}), -d(a^{p+1}))$.

The term cone comes from analogy with topology where for a continuous mapping $f : X \to Y$ one defines a topological space $C(f)$ as follows:

$$C(f) := \frac{Y \sqcup (X \times [0,1])}{f(x) \sim (y,0),\ (y,1) \sim (y',1)},$$

and this gives sequences such as

$$X \to Y \to C(f) \to \Sigma X$$

which in turn give rise to long exact sequences of homology groups.

Exercise 1.2. *1. There is a short exact sequence of complexes*

$$0 \to B^\bullet \to cone(f) \to A^\bullet \to 0.$$

2. If $f : A^\bullet \to B^\bullet$ is a monomorphism, then $cone(f)$ is homotopy equivalent to $coker(f)$.
3. If $f : A^\bullet \to B^\bullet$ is an epimorphism, then $cone(f)$ is homotopy equivalent to $ker(f)[1]$.
4. If $f : A[0] \to B[0]$ is a morphism of complexes situated in degree 0, then

$$cone(f) = [A \to B],$$

with B in degree 0 and with differential given by f.

Sequences isomorphic to

$$A^\bullet \to B^\bullet \to cone(f) \to A^\bullet[1]$$

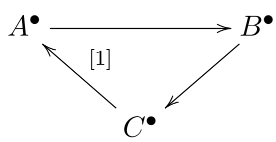
are called distinguished triangles. One of the properties of distinguished triangles is that if

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is distinguished, then

$$B^\bullet \to C^\bullet \to A^\bullet[1] \to B^\bullet[1]$$

is also distinguished. We can visualise distinguished triangles like



Given a triangulated category $\mathcal{C}$ we produce long exact sequce as follows.

Lemma 1.3. *Let $\mathcal{C}$ be a triangulated category. Then for any $U \in \mathcal{C}$ and any triangle*

$$A \to B \to C \to A[1]$$

the following sequences are exact:

$$\cdots \to Hom(U, A) \to Hom(U, B) \to Hom(U, C) \to Hom(U, A[1]) \ldots$$

$$\cdots \to Hom(C, U) \to Hom(B, U) \to Hom(A, U) \to Hom(C[-1], U) \to \ldots$$

2. Derived category of coherent sheaves and Fourier-Mukai transform

Let X be a smooth projective variety over a field. We define

$$D^b(X) := D^b_{coh}(\mathcal{O}_X - mod) \cong D^b_{coh}(QCoh(X)).$$

Let $f : X \to Y$ be a morphism of smooth projective varieties. Then $Lf^* : D^b(Y) \to D^b(X)$ and $Rf_* : D^b(X) \to D^b(Y)$ are well-defined adjoint functors. We also have an adjoint pair $\otimes^L$ and $\underline{RHom}$.

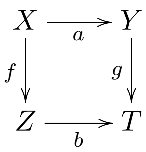
To simplify the notation we omit R and L from the notation from now on.

We will need the following two formulas:

(1) **Projection formula** Let $f : X \to Y$ be a morphism of smooth projective varieties. Then we have a natural isomorphism in $D^b(Y)$:

$$f_*(\mathcal{F} \otimes f^*(\mathcal{G})) \simeq f_*(\mathcal{F}) \otimes \mathcal{G}.$$

(2) **Base change formula** Given a cartesian diagram



with g (and thus also f) is flat. Then for any sheaf $\mathcal{F}$ on Z we have a natural isomorphism in $D^b(Y)$

$$g^* b_*(\mathcal{F}) \simeq a_* f^*(\mathcal{F}).$$

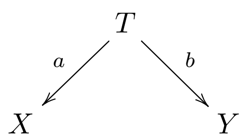
For any sheaf (so-called kernel) $\mathcal{F}$ on $X \times Y$ we may consider the Fourier-Mukai transform

$$FM_{\mathcal{F}} : D^b(Y) \to D^b(X)$$

given as the composition

$$\mathcal{E} \mapsto p_{1,*}(\mathcal{F} \otimes p_2^*(\mathcal{E})).$$

More generally we could define a Fourier-Mukai transform $FM_{T,\mathcal{G}} : D^b(Y) \to D^b(X)$ given any diagram



and an object $\mathcal{G} \in D^b(T)$ by the formula:

$$\Phi(\mathcal{E}) := a_*(\mathcal{G} \otimes b^*\mathcal{E}).$$

However the next lemma shows that we get essentially the same class of functors.

Lemma 2.1. *We have an isomorphism $FM_{T,\mathcal{G}} \simeq FM_{\mathcal{F}}$ where $\mathcal{F} = (a,b)_*(\mathcal{G})$.*

We list some properties of the Fourier-Mukai transform.

Proposition 2.2. *0. Given a morphism $f : X \to Y$ consider its graph $\Gamma \subset X \times Y$. We have*

$$FM_{\mathcal{O}_\Gamma} \simeq f^*$$
$$FM_{\mathcal{O}^t_\Gamma} \simeq f_*.$$

1. $FM_{\mathcal{F}} \circ FM_{\mathcal{G}} \simeq FM_{\mathcal{F}\circ\mathcal{G}}$ where

$$\mathcal{F} \circ \mathcal{G} := p_{13,*}(p_{12}^*\mathcal{F} \otimes p_{23}^*\mathcal{G})$$

2. $FM_{\mathcal{O}_\Delta} \simeq Id_{D^b(X)}$

3. $FM_{p_1^*(\mathcal{F}_1)\otimes p_2^*(\mathcal{F}_2)}(\mathcal{E}) \simeq \mathcal{F}_1 \otimes \Gamma(Y, \mathcal{F}_2 \otimes \mathcal{E})$

4. If

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F} \to \mathcal{F}'[1]$$

is a distinguished triangle, then for any $\mathcal{E} \in D^b(Y)$

$$FM_{\mathcal{F}'}(\mathcal{E}) \to FM_{\mathcal{F}}(\mathcal{E}) \to FM_{\mathcal{F}''}(\mathcal{E}) \to FM_{\mathcal{F}'}(\mathcal{E})[1]$$

is a distinguished triangle.

Proof. To prove (0) let $i = (id, f) : X \to X \times Y$ so that $\mathcal{O}_\Gamma \simeq i_*\mathcal{O}_X$. Now we compute

$$FM_{\mathcal{O}_\Gamma}(\mathcal{E}) = p_{1,*}(i_*(\mathcal{O}_X) \otimes p_2^*\mathcal{E}) \cong p_{1,*}(i_*(i^*p_2^*\mathcal{E})) \simeq (p_2 \circ i)^*(\mathcal{E}) \simeq f^*(\mathcal{E})$$

and

$$FM_{\mathcal{O}^t_\Gamma}(\mathcal{E}) = p_{2,*}(i_*(\mathcal{O}_X) \otimes p_1^*\mathcal{E}) \cong p_{2,*}(i_*(i^*p_1^*\mathcal{E})) \simeq (p_2 \circ i)_*(\mathcal{E}) \simeq f_*(\mathcal{E}).$$

To prove (1) we first show that

$$FM_{\mathcal{F}} \circ FM_{\mathcal{G}} \simeq FM_{X\times Y\times Z, p_{12}^*(\mathcal{F})\otimes p_{23}^*(\mathcal{G})}$$

(this uses the base change formula). Now from the Lemma above it follows that

$$FM_{X\times Y\times Z, p_{12}^*(\mathcal{F})\otimes p_{23}^*(\mathcal{G})} \simeq FM_{p_{13,*}(p_{12}^*(\mathcal{F})\otimes p_{23}^*(\mathcal{G}))} = FM_{\mathcal{F}\circ\mathcal{G}}.$$

(2) follows from (0) with $f = id$.

In (3) we use the isomorphism $FM_{\mathcal{O}_{X\times Y}}(\mathcal{E}) \simeq \mathcal{O}_X \otimes \Gamma_Y(\mathcal{E})$ which follows from the base change.

Finally (4) is obvious because all the functors involved are triangulated. □

3. Semiorthogonal decompositions

Let $\mathcal{C}$ be a triangulated category, and let $\mathcal{A}$ be its full subcategory.

Definition 3.1. The right orthogonal to $\mathcal{A}$ is defined as:

$$\mathcal{A}^\perp = \{C \in \mathcal{C} : Hom(A, C) = 0 \ \forall A \in \mathcal{A}\}.$$

Similarly the left orthogonal to $\mathcal{A}$ is defined as:

$$^\perp\mathcal{A} = \{C \in \mathcal{B} : Hom(C, A) = 0 \ \forall A \in \mathcal{A}\}.$$

Example 3.2. Let $\mathcal{C} = D^b(X)$, $\mathcal{A} = \langle \mathcal{O}_X \rangle$. Then the right orthogonal to $\mathcal{A}$ consists of Γ_X-acyclic compexes.

Lemma 3.3. *Let $\mathcal{A}, \mathcal{B}$ be triangulated subcategories of $\mathcal{C}$ with the inclusion functors denoted by i and j respectively. Assume that $Hom(B, A) = 0$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$ as a triangulated category*
2. *For each $X \in \mathcal{C}$ there exists a distinguished triangle*
$$B \to X \to A \to B[1]$$
with $A \in \mathcal{A}$, $B \in \mathcal{B}$.
3. *$\mathcal{B} = {}^{\perp}\mathcal{A}$ and there exists a functor $i^* : \mathcal{C} \to \mathcal{A}$ which is left adjoint to $i : \mathcal{A} \to \mathcal{C}$*
4. *$\mathcal{A} = \mathcal{B}^{\perp}$ and there exists a functor $j^! : \mathcal{C} \to \mathcal{B}$ which is right adjoint to $j : \mathcal{B} \to \mathcal{C}$*

If these conditions are satisfied $\mathcal{A}$ is called left admissible, $\mathcal{B}$ is called right admissible and we say that we have a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}, \mathcal{B} \rangle$. In this case the components A, B of X in 2. are defined uniquely up to an isomorphism.

Proof. It is obvious that (2.) implies (1.). The reverse implications is proved as follows. Consider the subcategory $\mathcal{C}' \subset \mathcal{C}$ consisting of objects X that admit a decomposition into distinguished triangle as in (2.). $\mathcal{C}'$ contains both $\mathcal{A}$ and $\mathcal{B}$ and thus it is sufficient to prove that $\mathcal{C}'$ is triangulated. $\mathcal{C}'$ is obviously closed under shifts. The fact that $\mathcal{C}'$ is closed under cones is the generalized octahedron axiom giving a diagram of distinguished triangles:

$$\begin{array}{ccccc} B & \longrightarrow & X & \longrightarrow & A \\ {\scriptstyle f_B}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f_A}\downarrow \\ B' & \longrightarrow & X' & \longrightarrow & A' \\ \downarrow & & \downarrow & & \downarrow \\ cone(f_B) & \longrightarrow & cone(f) & \longrightarrow & cone(f_A) \end{array}$$

It is easy to see that (4.) implies (2.). Indeed the required triangle is
$$j^! X \to X \to cone(j^! X \to X)$$
with $Hom(B, X) \cong Hom(B, j^! X)$ implying $A = cone(j^! X \to X) \in \mathcal{A} = \mathcal{B}^{\perp}$.

Let us prove that (2.) implies (4.) We first note that in any diagram

$$\begin{array}{ccccc} B & \longrightarrow & X & \longrightarrow & C \\ \vdots & & \downarrow & & \vdots \\ B' & \longrightarrow & X' & \longrightarrow & C' \end{array}$$

the dotted arrows uniquely exist. Therefore the assignment $X \mapsto i^! X := B$ is functorial once we fix a B for each X.

Similary, (3.) is equivalent to (2.)

□

Example 3.4. Let $\mathcal{C} = D^b(X)$ and $\mathcal{A}$ be the triangulated category generated by skyscraper sheaves $\mathcal{O}_x, x \in X$.

Then $\mathcal{A}$ is neither right nor left admissible. Indeed both orthogonals to $\mathcal{A}$ are zero, whereas $\mathcal{A} \neq \mathcal{C}$.

Example 3.5. We will use the Serre functor in the next lecture to show that a variety X with $K_X = 0$ has no nontrivial admissible subcategories in $D^b(X)$.

Remark 3.6. Can we have fully orthogonal decompositions of $D^b(X)$? No, unless X is disconnected. Indeed if $D^b(X) = \mathcal{A} \times \mathcal{B}$, then we can write $\mathcal{O}_X$ as a direct sum of two sheaves which implies that X is reducible.

Definition 3.7. If $\mathcal{A}_1, \dots, \mathcal{A}_r$ are triangulated subcategories of $\mathcal{C}$ $Hom(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $j > i$ and $\mathcal{A}_i$ generate $\mathcal{C}$, then we say that $\mathcal{C}$ admits a semi-orthogonal decomposition $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$.

Saturatedness?

4. Exceptional collections

Definition 4.1. An object $E \in \mathcal{C}$ is called exceptional if $Hom^*(E, E) = \mathbb{C}[0]$.

Example 4.2. The structure sheaf $\mathcal{O}_X \in D^b(X)$ is exceptional if and only if $h^{i,0}(X) = 0$ for $i > 0$. In this case any line bundle L on X is exceptional.

Definition 4.3. A sequence $E_1, \dots, E_r$ of exceptional objects is called an exceptional sequence if $Hom^*(E_j, E_i) = 0$ for $j > i$.

An exceptional collection is called full if it generates $\mathcal{C}$, or equivalently, $\langle \langle E_1 \rangle, \dots, \langle E_r \rangle \rangle$ is a semi-orthogonal decomposition of $\mathcal{C}$.

Proposition 4.4. *Let $E_1, \dots E_r$ be an exceptional collection of sheaves on X. Assume that there exists a resolution of the diagonal on $X \times X$ of the form*

$$0 \to p_1^* E_1 \otimes p_2^* F_1 \to \dots \to p_1^* E_r \otimes p_2^* F_r \to \mathcal{O}_\Delta \to 0$$

where F_i, $i = 1, \dots, r$ is also a sequence of sheaves on X (not necessarily exceptional). Then the collection E_i is full.

Proof. The resolution above is equivalent to a list of short exact sequences:

$$0 \to H_{k-1} \to p_1^* E_k \otimes p_2^* F_k \to H_k \to 0, \; k = 1, \dots, r$$

$$H_0 = 0, \; H_r = \mathcal{O}_\Delta$$

We prove the following statement by induction on k:

$$FM_{H_k}(\mathcal{C}) \subset \langle E_1, \dots, E_k \rangle .$$

Indeed, the statement is trivial for $k = 0$ and the distinguished triangle

$$FM_{H_{k-1}}(\mathcal{E}) \to E_k \otimes \Gamma(F_k \otimes \mathcal{E}) \to FM_{H_k}(\mathcal{E}) \to FM_{H_{k-1}}(\mathcal{E})[1]$$

is used to make the induction step. Substituing $k = r$ gives the result. □

Theorem 4.5. *The sequences $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}$ and $\Omega^n(n), \dots, \Omega^1(1), \mathcal{O}$ are full exceptional collections on $\mathbb{P}^n$.*

It is well-known that the appropriate cohomology groups vanish so that the first sequence is exceptional. The second sequence can be seen to be exceptional by induction using the exact sequence:

$$0 \to \Omega^1 \to V^*(1) \to \mathcal{O} \to 0$$

and its exterior powers

$$0 \to \Omega^k \to \wedge^k V^*(k) \to \Omega^{k-1} \to 0.$$

Lemma 4.6. *We have the following resolution:*

$$0 \to p_1^*(\Omega^n(n)) \otimes p_2^*(\mathcal{O}(-n)) \to \cdots \to p_1^*(\Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1)) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

Proof. It suffices to find a regular section $s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*(T(-1)) \otimes p_2^*(\mathcal{O}(1)))$ which vanishes precisely on the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. Indeed, in this case the Koszul complex gives a resolution of the diagonal as required. Indeed if $\mathcal{E} = p_1^*(\Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1))$, we may consider the following complex:

$$0 \to \wedge^n \mathcal{E} \to \cdots \wedge^2 \mathcal{E} \to \mathcal{E} \to \mathcal{O}_\Delta \to 0$$

where each differential is a contraction with $s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{E}^\vee)$. The standard theorem of homological algebra says that the complex above is exact if the section s is regular (that is, generic in a certain sense).

To find such a regular section s we start with the Euler exact sequence on $\mathbb{P}^n$:

$$0 \to \mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to T(-1) \to 0$$

and then consider the following composition:

$$\phi : p_2^*\mathcal{O}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to p_1^*T(-1).$$

At the point $(l_1, l_2) \in \mathbb{P}^n \times \mathbb{P}^n$ we have

$$\phi_{(l_1,l_2)} : l_1 \subset V \to V/l_1.$$

In particular, $\phi_{(l_1,l_2)} = 0$ if and only if $l_1 = l_2$.

Let $s \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, p_1^*(T(-1)) \otimes p_2^*\mathcal{O}(1))$ be the section corresponding to ϕ. Then s vanishes precisely along the diagonal and s is regular since of the codimension of the diagonal is equal to n which is the rank of the bundle. □