DERIVED CATEGORIES: LECTURE 3

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References

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1. Generators and strong generators

Definition 1.1. $T \in \mathcal{C}$ is called a classical generator if the smallest thick[1] triangulated subcategory which contains T is $\mathcal{C}$. T is called a strong generated if there exist an integer n such that any object $C \in \mathcal{C}$ can be obtain starting with T using direct sums, direct summands and at most n cones.

Example 1.2. If $\mathcal{C}$ has a full exceptional collection $E_1, \dots, E_r$, then $E_1 \oplus \dots \oplus E_r$ is a strong generator.

Theorem 1.3. *1. If $\mathcal{O}_X(1)$ is a very ample line bundle on X, then $\bigoplus_{j=0}^n \mathcal{O}(-j)$, $n = dim(X)$ is a generator of $D^b(X)$.*

2. Any classical generator of $D^b(X)$ is strong.

3. If $\mathcal{C}$ admits a strong (resp. classical) generator, then any left or right admissible subcategory $\mathcal{A} \subset \mathcal{C}$ also admits a strong (resp. classical) generator.

Proof. (1) Let $i : X \to \mathbb{P}^N$ be a projective embedding satisfying $\mathcal{O}_X(1) \simeq i^*\mathcal{O}_{\mathbb{P}^N}(1)$. Let $\mathcal{C} \subset D^b(X)$ be the smallest thick triangulated subcategory containing $\mathcal{O}_X(k), k = -n, \dots, 0$.

We first prove that $\mathcal{C}$ contains all $\mathcal{O}_X(k)$, $k \leq 0$. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^N}(-1)^{N+1}$ and $s \in \Gamma(\mathbb{P}^N, \mathcal{E}^\vee)$ be a nonvanishing section. Since $Z(s) = 0$, the Koszul compex corresponding to s gives rise to exact sequence of sheaves

$$0 \to \Lambda^{N+1}\mathcal{E} \to \dots \to \Lambda^2\mathcal{E} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^N} \to 0.$$

[1]Thick means triangulated and closed under taking direct summands.

We restrict this sequence to X and consider the truncation

$$\mathcal{V}^\bullet = [\Lambda^{n+1}\mathcal{E}\big|_X \to \cdots \to \Lambda^2\mathcal{E}\big|_X \to \mathcal{E}\big|_X] =$$
$$= [\Lambda^{n+1}k^{N+1} \otimes \mathcal{O}_X(-n-1) \to \cdots \to \Lambda^2 k^{N+1} \otimes \mathcal{O}_X(-2) \to k^{N+1} \otimes \mathcal{O}_X(-1)].$$

Let $\mathcal{H} = Ker(\Lambda^{n+1}\mathcal{E} \to \Lambda^n\mathcal{E})$. $\mathcal{H}[n]$ is a subcomplex in $\mathcal{V}^\bullet$ and the quotient is quasi-isomorphic to $\mathcal{O}_X$, thus we have a distinguished triangle:

$$\mathcal{H}[n] \to \mathcal{V}^\bullet \to \mathcal{O}_X \to \mathcal{H}[n+1].$$

Since $n = dim(X)$, we have $H^{n+1}(X, \mathcal{H}) = 0$, thus the third morphism in the distinguished triangle is zero, and it splits, from which we see, that $\mathcal{O}_X$ is a direct summand in $\mathcal{V}^\bullet$. Therefore, $\mathcal{O}_X(1)$ is a direct summand in $\mathcal{V}^\bullet \otimes \mathcal{O}_X(1) \in \mathcal{C}$, thus $\mathcal{O}_X(1) \in \mathcal{C}$ and by induction we obtain that $\mathcal{O}_X(k) \in \mathcal{C}$ for all $k \geq -n$.

Similarly, using the truncation

$$[\Lambda^N k^{N+1} \otimes \mathcal{O}_X(-N) \to \cdots \to \Lambda^{N-d+1}k^{N+1} \otimes \mathcal{O}_X(-N+d-1)]$$

and its twists one shows that $\mathcal{O}_X(k) \in \mathcal{C}$ for all $k < -n$ as well.

Finally will prove that $\mathcal{C}$ contains all coherent sheaves, thus $\mathcal{C} = D^b(X)$ and $L^{\otimes -n} \oplus \cdots \oplus L^{\otimes -1} \oplus \mathcal{O}$ is a generator.

Any coherent sheaf $\mathcal{F}$ on $X \subset \mathbb{P}^N$ admits an infinite resolution by direct sums of the object $\mathcal{O}(k)$. Truncating this complex as above shows that $\mathcal{F}$ lies is $\mathcal{C}$.

(2.) One first proves that if T is a classical generator of $D^b(X)$, then $T \boxtimes T$ is a classical generator of $D^b(X \times X)$. Now $\mathcal{O}_{\Delta_X} \in D^b(X \times X)$ is generated by $T \boxtimes T$ using a finite number N of cones, thus the same holds true for $Id_X = FM_{\mathcal{O}_{\Delta_X}}$ in terms of $FM_{T \boxtimes T} = T \otimes R\Gamma(T \otimes \bullet)$. Therefore, any $\mathcal{F}$ is generated by T using N cones.

(3.) If $\mathcal{A} \subset \mathcal{C}$ be right admissible and if T is a strong (resp. classical) generator of $\mathcal{C}$, then $i^!(T)$ is a strong (resp. classical) generator of $\mathcal{A}$.

□

Remark 1.4. One of the important consequences of having a strong generator is that by a theorem of Keller if T is a generator for $D^b(X)$, then one has

$$D^b(X) \cong D^b_{perf}(mod - A),$$

where on the right we have the derived category of perfect dg-modules over dg-algebra $A = RHom(T, T)$. The equivalence sends a complex $\mathcal{F}^\bullet$ to the right A-module $RHom(T, \mathcal{F}^\bullet)$. We can think of this equivalence as a derived equivalence between X and a non-commutative derived affine variety corresponding to A.

Example 1.5. Let $T = \mathcal{O} \oplus \mathcal{O}(1)$ be the generator of $\mathbb{P}^1 = \mathbb{P}(V)$. Let x_1, x_2 be a basis of $H^0(\mathbb{P}^1, \mathcal{O}(1))$. Then

$$A = RHom(T, T) \cong k \cdot e_0 \oplus k \cdot x_1 \oplus k \cdot x_2 \oplus k \cdot e_1,$$

with all mutliplications vanishing except for

$$e_0^2 = e_0;\ e_1^2 = e_1;\ e_1 x_i e_0 = x_i,\ i = 1, 2.$$

Let U be an A-module. Let $U_0 = Im(e_0)$, $U_1 = Im(e_1)$. We have $U = U_0 \oplus U_1$ and x_1, x_2 give rise to morphisms $U_0 \to U_1$. Such data is by definition the same thing as a representation of a quiver $S_2 : \bullet \rightrightarrows \bullet$.

More generally any exceptional collection without higher Ext's in $D^b(X)$ gives rise to an equivalence between $D^b(X)$ and the derived category of representations of a quiver with relations.

2. Saturatedness

Definition 2.1. A contravariant (resp. covariant) functor $H : \mathcal{C} \to Vect/k$ is called a cohomological functor if for any triangle

$$X \to Y \to Z \to X[1]$$

we have a long exact sequence

$$\cdots \to H(X[1]) \to H(Z) \to H(Y) \to H(X) \to H(Z[-1]) \to \ldots.$$

(resp.

$$\cdots \to H(Z[-1]) \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to \ldots).$$

H is called of finite type if $\bigoplus_{k\in\mathbb{Z}} H(X[k])$ are finite-dimensional vector spaces for all $X \in \mathcal{C}$.

Definition 2.2. A triangualted category $\mathcal{C}$ of finite type is called right (resp. left) saturated if any contravariant (resp. covariant) cohomological functor of finite type

$$F : \mathcal{C} \to D^b(Vect/k)$$

is representable. $\mathcal{C}$ is called saturated if it is left and right saturated.

Proposition 2.3. *1. If $\mathcal{A}$ is right (resp. left) saturated, for any triangulated category of finite type $\mathcal{C}$ any fully faithful embedding $\mathcal{A} \subset \mathcal{C}$, $\mathcal{A}$ is right (resp. left) admissible.*

2. If $\mathcal{A}$ is saturated, then $\mathcal{A}$ admits a Serre functor.

3. If $\mathcal{C}$ is saturated and $\mathcal{A} \subset \mathcal{C}$ is left (or right) admissble, then $\mathcal{A}$ is saturated.

4. If $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semi-orthogonal decomposition and $\mathcal{A}$ and $\mathcal{B}$ are saturated, then $\mathcal{C}$ is saturated.

Proof. (1.) We prove that if $\mathcal{A}$ is right saturated, then $i : \mathcal{A} \to \mathcal{C}$ admits a right adjoint, thus $\mathcal{A}$ will be right admissible. For any $C \in \mathcal{C}$ consider a functor

$$F_C(A) = Hom_{\mathcal{C}}(i(A), C).$$

This functor being a contravariant cohomological functor of finite type is representable by some object which we denote $i^!(C)$:

$$F_C(A) = Hom_{\mathcal{A}}(A, i^!(C)).$$

Any choice $\{i^!(C)\}_{C\in\mathcal{C}}$ will be functorial in $\mathcal{C}$ and by construction the functor $i^!$ is right adjoint to i.

(2.) We have shown last time that $\mathcal{C}$ admits a Serre functor if the functors $Hom(C, \bullet)^*$ and $Hom(\bullet, C)^*$ are representable for all $C \in \mathcal{C}$. Since both these kinds of functors are cohomological of finite type, it follows from saturatedness that they are representable.

(3.) We first prove that if $\mathcal{C}$ is right saturated and $\mathcal{A}$ is right admissible, then $\mathcal{A}$ is right saturated. Let $H : \mathcal{A} \to Vect$ be a contravariant cohomological functor of finite type. Consider a functor

$$H' = H \circ i^! : \mathcal{C} \to Vect.$$

There is an object $C \in \mathcal{C}$ such that
$$H' \cong Hom(\bullet, C).$$
Now
$$H \cong H' \circ i \cong Hom(i(\bullet), C) \cong Hom(\bullet, i^!(C)),$$
thus $i^!(C) \in \mathcal{A}$ represents H.

We now prove that if $\mathcal{C}$ is right saturated and $\mathcal{A}$ is left admissible, then $\mathcal{A}$ is right saturated. Let again $H : \mathcal{A} \to Vect$ be a contravariant cohomological functor of finite type. Consider a functor
$$H' = H \circ i^* : \mathcal{C} \to Vect.$$
There is an object $C \in \mathcal{C}$ such that
$$H' \cong Hom(\bullet, C).$$
We claim that $C \in \mathcal{A}$. Indeed for any $B \in {}^{\perp}\mathcal{A}$
$$Hom(B, C) \cong H'(B) = H(i^*B) = 0,$$
thus $B \in \mathcal{A}$. Since the representing object C lies in $\mathcal{A}$ it also represents the restriction $H \simeq H'\big|_{\mathcal{A}}$.

(4.) We omit the proof. See [BK]. □

Theorem 2.4. *If $\mathcal{C}$ admits a strong generator, then $\mathcal{C}$ is saturated.*

This is the one of the main results of [BvdB]. The proof of this theorem is complicated and we omit it.

3. Semi-orthogonal decomposition of fibre bundles

Theorem 3.1. *Let E be a vector bundle of rank $n+1$ over X. Let $p : \mathbb{P}(E) \to X$ be the corresponding projective bundle and let $\mathcal{O}(1)$ be the canonical line bundle on $\mathbb{P}(E)$. Then $p^* : D^b(X) \to D^b(\mathbb{P}(E))$ is a fully faithful embedding and there is a semi-orthogonal decomposition*
$$D^b(\mathbb{P}(E)) = \left\langle p^*D^b(X), p^*D^b(X) \otimes \mathcal{O}(1) \ldots, p^*D^b(X) \otimes \mathcal{O}(n) \right\rangle.$$
In particular if X admits a full exceptional collection, then $\mathbb{P}(E)$ admits a full exceptional collection.

This theorem of Orlov is a particular case of a more general theorem proved later by Samokhin:

Theorem 3.2. *Let $p : Y \to X$ be a flat morphism of smooth projective varieties. Assume that there exist a sequence of vector bundles $\mathcal{F}_1, \ldots, \mathcal{F}_r$ on $\mathcal{X}$ such that the restrictions $\mathcal{F}_{1,x}, \ldots, \mathcal{F}_{r,x}$ of this sequence to each fiber Y_x, $x \in X$ give a full exceptional collection. Then $\Phi_i : D^b(X) \to D^b(Y)$, $\Phi_i(A) := p^*(A) \otimes \mathcal{F}_i$ is a fully faithful embedding and there is a semi-orthogonal decomposition*
$$D^b(Y) = \left\langle p^*D^b(X) \otimes \mathcal{F}_1, p^*D^b(X) \otimes \mathcal{F}_2 \ldots, p^*D^b(X) \otimes \mathcal{F}_r \right\rangle.$$

Proof. The proof goes in following steps.

Step 1. Φ_i is a fully faithful embedding for each $i = 1, \ldots, r$. We compute

$$\begin{aligned} Hom(p^*(A) \otimes \mathcal{F}_i, p^*(B) \otimes \mathcal{F}_i) &\cong Hom(p^*(A), p^*(B) \otimes \underline{Hom}(\mathcal{F}_i, \mathcal{F}_i)) \\ &\cong Hom(A, B \otimes p_* \underline{Hom}(\mathcal{F}_i, \mathcal{F}_i)). \end{aligned}$$

We now that prove that $p_* \underline{Hom}(\mathcal{F}_i, \mathcal{F}_i) \simeq \mathcal{O}_X$ (recall that our p_* is the derived functor). This follows from the base change and the fact that the restrictions of $\mathcal{F}_i$ to all fibers are exceptional.

Step 2. $Hom(p^* A \otimes \mathcal{F}_j, p^* B \otimes \mathcal{F}_i) = 0$ for $j > i$, that is the sequence of subcategories $p^* D^b(X) \otimes \mathcal{F}_1, \ldots, p^* D^b(X) \otimes \mathcal{F}_r$ is semi-orthogonal. This is step is very similar to step 1.

Step 3. The subcategory $\mathcal{A} = \langle p^* D^b(X) \otimes \mathcal{F}_1, \ldots, p^* D^b(X) \otimes \mathcal{F}_r \rangle \subset D^b(X)$ is admissible. Indeed $\mathcal{A}$ is saturated, hence admissible.

Step 4. $\mathcal{A}$ contains all $k(y), y \in Y$, hence the orthogonals to $\mathcal{A}$ vanish and $\mathcal{A}$ coincides with $D^b(X)$.

For more details see [S].

□

Corollary 3.3. *Let X, Y be smooth projective varieties admitting full exceptional collections $E_1, \ldots, E_r$ and $F_1, \ldots, F_l$ respectively. Then $E_i \boxtimes F_j$ is a full exceptional collection on $X \times Y$ (we allow any ordering of $\{E_i \boxtimes F_j\}$ which is compatible with the ordering of $\{E_i\}$ and the ordering of $\{F_j\}$).*

Example 3.4. $\{\mathcal{O}(i,j) := \mathcal{O}(i) \otimes \mathcal{O}(j),\ 0 \leq i, j \leq n\}$ is a full exceptional collection on $\mathbb{P}^n \times \mathbb{P}^n$.

4. Semi-orthogonal decomposition of blow ups

Theorem 4.1. *Let $\widetilde{X}$ be a blow up of a smooth projective variety X along a smooth subvariety $Y \subset X$. Let $\widetilde{Y}$ be the exceptional divisor:*

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\ j\ } & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \longrightarrow & X \end{array}$$

Recall that $p : \widetilde{Y} \to Y$ is a projective bundle and let $\mathcal{O}(1)$ be the corresponding canonical line bundle on $\widetilde{Y}$. Then

1. $\pi^* : D^b(X) \to D^b(\widetilde{X})$ is a fully faithful embedding.

2. $j_* : D^b(\widetilde{Y}) \to D^b(\widetilde{X})$ restricted to each $p^* D^b(Y) \otimes \mathcal{O}(k)$, $k \in \mathbb{Z}$ is a fully faithful embedding.

3. There is a semi-orthogonal decomposition

$$D^b(\widetilde{X}) = \left\langle D^b(Y)_{c-1}, \ldots, D^b(Y)_1, D^b(X)_0 \right\rangle.$$

Here $D^b(X)_0 = \pi^* D^b(X)$, $D^b(Y)_k = j_* p^* D^b(Y) \otimes \mathcal{O}(-k)$ and $c = codim(Y/X)$.

In particular, if X and Y admit full exceptional collections, then $\widetilde{X}$ also has one.

In order to prove semi-orthogonality we will use the following Lemma.

Lemma 4.2. *Let $j : D \to X$ be an embedding of a smooth divisor. Then for any object $\mathcal{F} \in D^b(D)$ there is a following triangle in $D^b(D)$:*

$$\mathcal{F} \otimes \mathcal{O}(-D) \to \mathcal{F} \to j^* j_* \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}(-D)[1].$$

Proof of the Theorem. (1) follows from the fact that $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and adjunction between π^*, π_*.

We will now check (2) and semi-orthogonality in (3). We use adjunctions and the Lemma with $\mathcal{O}(-D) = \mathcal{O}(-\widetilde{Y}) = \mathcal{O}(1)$:

$$\begin{aligned} Hom(j_*(p^*\mathcal{F}_1 \otimes \mathcal{O}(k)), j_*(p^*\mathcal{F}_2 \otimes \mathcal{O}(k))) &= Hom(j^* j_*(p^*\mathcal{F}_1 \otimes \mathcal{O}(k)), p^*\mathcal{F}_2 \otimes \mathcal{O}(k)) = \\ &= Hom(p^*\mathcal{F}_1 \otimes \mathcal{O}(k), p^*\mathcal{F}_2 \otimes \mathcal{O}(k)) = \\ &= Hom(\mathcal{F}_1, \mathcal{F}_2). \end{aligned}$$

Similarly, if $k > i$, then

$$\begin{aligned} Hom(j_*(p^*\mathcal{F}_1 \otimes \mathcal{O}(k)), j_*(p^*\mathcal{F}_2 \otimes \mathcal{O}(i))) &= Hom(j^* j_*(p^*\mathcal{F}_1 \otimes \mathcal{O}(k)), p^*\mathcal{F}_2 \otimes \mathcal{O}(i)) = \\ &= Hom(p^*\mathcal{F}_1 \otimes \mathcal{O}(k), p^*\mathcal{F}_2 \otimes \mathcal{O}(i)) = 0. \end{aligned}$$

Finally, we have

$$\begin{aligned} Hom(\pi^*\mathcal{G}, j_*(p^*\mathcal{F} \otimes \mathcal{O}(k))) &= Hom(\mathcal{G}, \pi_* j_*(p^*\mathcal{F} \otimes \mathcal{O}(k))) = \\ &= Hom(\mathcal{G}, i_* p_*(p^*\mathcal{F} \otimes \mathcal{O}(k))) = \\ &= Hom(\mathcal{G}, i_*(\mathcal{F} \otimes p_*(\mathcal{O}(k))) = 0 \end{aligned}$$

since $p_*(\mathcal{O}(k)) = 0$ for $k = -1, \ldots, -c+1$.

We omit the proof of fullness. □